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Existence of Global Strong Solutions to a Beam–Fluid Interaction System

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Abstract

We study an unsteady nonlinear fluid–structure interaction problem which is a simplified model to describe blood flow through viscoelastic arteries. We consider a Newtonian incompressible two-dimensional flow described by the Navier–Stokes equations set in an unknown domain depending on the displacement of a structure, which itself satisfies a linear viscoelastic beam equation. The fluid and the structure are fully coupled via interface conditions prescribing the continuity of the velocities at the fluid–structure interface and the action–reaction principle. We prove that strong solutions to this problem are global-in-time. We obtain, in particular that contact between the viscoelastic wall and the bottom of the fluid cavity does not occur in finite time. To our knowledge, this is the first occurrence of a no-contact result, and of the existence of strong solutions globally in time, in the frame of interactions between a viscous fluid and a deformable structure.

1. Introduction

In this paper, we focus on the interactions between a viscous incompressible Newtonian fluid and a moving viscoelastic structure located on one part of the fluid domain boundary. This work is motivated by the study of blood flow in arteries and the fluid–structure interaction (FSI) model we consider here can be viewed as a simplified version of a standard model/benchmark for FSI problems/solvers in hemodynamics [13,42]. Here we are interested in global existence results and the possibility of the collapse of the arterial wall. Consequently, we investigate whether or not a collision occurs between the moving boundary and the bottom of the fluid cavity together with the existence of global-in-time strong solutions.

A vast majority of studies on the existence of a solution for fluid–structure interaction problems concerns the motion of a rigid solid in a viscous incompressible Newtonian fluid, whose behavior is then described by the Navier–Stokes equations (see [6,10,11,14,26,32,33,35,44,45,47,48]). A challenging point is the possibility of body–body or body–boundary collisions. In particular, these existence results are valid up to contact, except those of [44] and [14], where special weak solutions after possible collisions are built in the two dimensional case and in the three dimensional case respectively. The contact issue is further investigated in [28] and [29] where a no-collision result is proven in a bounded two-dimensional cavity. The three-dimensional situation is also explored in [31] and in [30]. We mention that, in [44, 46], collisions, if any, are proved to occur with zero relative *translational* velocity as soon as the boundaries of the rigid bodies are smooth enough and the gradient of the surrounding fluid velocity field is square integrable. A complementary study of the influence of the smoothness of boundaries on the existence of collisions has been recently tackled in [17] and [18]. In [17] it is proven that below a critical regularity of a two dimensional rigid body boundary, namely $C^{1,1/2}$, collisions may occur; in [19] and [20] slip boundary conditions at the fluid–structure interface are introduced and the existence of weak solutions up to collision/existence of finite time collisions are proven respectively.

Fewer studies consider the case of an elastic structure evolving in a viscous incompressible Newtonian flow. We refer the reader to [12] and [4] where the structure is described by a finite number of eigenmodes or to [3] for an artificially damped elastic structure while, for the case of a three-dimensional elastic structure interacting with a three-dimensional fluid, we mention [16,24] in the steady-state case and [7,8,34,43] for the full unsteady case. In the latter, the authors consider the existence of strong solutions for small enough data locally in time. In this field, the question of selfcontact and/or body–boundary collisions remains entirely open, to our knowledge.

Concerning the fluid–beam, or more generally fluid–shell coupled systems that we consider herein, the two dimensional/one dimensional steady state case is considered in [25] for homogeneous Dirichlet boundary conditions on the fluid boundaries (that are not the fluid–structure interface). The existence of a unique strong enough solution is obtained for small enough applied forces. In the unsteady framework we refer to [5] where a three dimensional/two dimensional fluid–plate coupled system is studied and where the structure is a damped plate in flexion. The case of an undamped plate is studied in [23]. The previous results deal with the existence of weak solutions, that is in the energy spaces, and rely on the only tranversal motion of the elastic beam that enables to circumvent the lack of regularity of the fluid domain boundary (that is not even Lipschitz). These results also apply to a two dimensional/one dimensional fluid–shell coupled problem which is recently considered in [41]. In this reference, the authors give an alternative proof of the existence of weak solutions based on ideas coming from numerical schemes [21]. The existence of strong solutions for three dimensional/two dimensional, or two dimensional/one dimensional coupled problems involving a damped elastic structure are studied in [1,38,39]. The proofs of [38,39] are based on a splitting strategy for the Stokes system and on an implicit treatment of the *so called* fluid added mass effect. Finally, the coupling of a three dimensional Newtonian fluid and a linearly elastic Koiter shell is recently studied in [37]. In this study, the mid-surface of the structure is not flat anymore and the existence of weak solutions is obtained.

In all the results we mentioned up to now, the existence of strong solutions is obtained locally in time while the existence of weak solutions is obtained up to contact between the elastic structure and the fluid cavity. Consequently, if one wants to prove the existence of solutions globally in time, either one should give a sense to solutions in case of collision or one should prove that no collision occurs. In this paper we investigate these issues on a two dimensional/one dimensional fluid–beam coupled problem in which the beam is viscoelastic and moves only in the tranversal direction relatively to its flat mid-surface. One already knows that a unique strong solution exists locally in time [39], whereas weak solutions exist as long as the beam does not touch the bottom of the fluid cavity (see [5] or [23] and [41] in the undamped beam case). Note that, in this model, the fluid domain is a subgraph whose regularity depends on the structure displacement. In connection with the rigid-body case, we mention that energy estimates ensure that the beam displacement belongs to $L^\infty_t(H^2_x)$ that embeds in $L^\infty_t(C^{1,\alpha}_x)$, for $\alpha = 1/2$. This corresponds precisely to the threshold exhibited in [17]. In this paper, the strategy we develop is first to prove that no contact occurs and next, to propagate the solution regularity. In the second step, the cornerstone is an elliptic regularity result for the inhomogeneous Stokes system valid for nonstandard regularity of the domain boundary.

2. General Setting, Main Result and Formal Argument

We consider a two dimensional container whose boundary is made of a one dimensional viscoelastic beam, which is a simplified linear viscolelastic Koiter shell model [41]. Due to the complexity of the underlying fluid–structure interaction problem we assume that only the upper part of the fluid cavity is moving. The fluid domain denoted $\mathcal{F}(t) \subset \mathbb{R}^2$ depends then on time since it depends on the structure displacement η. It reads

$$\mathcal{F}(t) := \{(x, y) \in \mathbb{R}^2,\ x \in (0, L),\ y \in (0, h(x, t))\}$$

where $(x, t) \mapsto h(x, t) = 1 + \eta(x, t)$ stands for the "deformation" of the beam. We assume that the fluid is two dimensional, homogeneous, viscous, incompressible and Newtonian. Its velocity-field u and internal pressure p satisfy the incompressible Navier–Stokes equations in $\mathcal{F}(t)$:

$$\rho_f(\partial_t u + u \cdot \nabla u) - \operatorname{div} \sigma(u, p) = 0, \tag{1}$$

$$\operatorname{div} u = 0. \tag{2}$$

The fluid stress tensor $\sigma(u, p)$ is given by the Newtonian law:

$$\sigma(u, p) = \mu(\nabla u + \nabla u^\top) - p\mathrm{I}_2.$$

Here μ denotes the viscosity of the fluid and ρ_f its density. The structure motion is given by a linear damped beam equation in flexion:

$$\rho_s \partial_{tt} h - \beta \partial_{xx} h + \alpha\, \partial_{xxxx} h - \gamma\, \partial_{xxt} h = \phi(u, p, h), \quad \text{on} \quad (0, L), \tag{3}$$

where α, β, γ are positive given constants and $\rho_s > 0$ denotes the structure density.

We emphasize that the beam equation is set in a reference configuration whereas the fluid equations are written in Eulerian coordinates and consequently in an unknown domain. Note moreover, that we choose to write the beam equation on h and not on the beam displacement $\eta = h - 1$ as is standard, since this is equivalent in the case one considers here and since it simplifies the presentation. The fluid and beam equations are coupled through the source term $\phi(u, p, h)$ in (3), which corresponds to the trace of the second component of $\sigma(u, p)n\mathrm{d}l$ transported in the beam reference configuration. The coupling term can be written as:

$$\phi(u, p, h)(x, t) = -e_2 \cdot \sigma(u, p)(x, h(x, t), t)(-\partial_x h(x, t)\, e_1 + e_2), \tag{4}$$

for $(x, t) \in (0, L) \times (0, T)$, where (e_1, e_2) denotes the canonical basis of $\mathbb{R}^2$. The fluid and the beam are coupled also through the kinematic condition

$$u(x, h(x, t), t) = \partial_t h(x, t) e_2, \quad (x, t) \in (0, L) \times (0, T). \tag{5}$$

We complement our system with the following conditions on the remaining boundaries of the container:

– L-periodicity with respect to x for the fluid and the beam;
– no-slip boundary conditions on the bottom of the fluid container:

$$u(x, 0, t) = 0. \tag{6}$$

Note that since the question of contact is mostly a local one we assume periodic boundary conditions at the outlet for the fluid and replace the standard assumptions of “clamped” arterial wall also by periodic boundary conditions for the beam.

An important remark on this coupled system is that the incompressibility condition together with boundary conditions imply:

$$\int_0^L \partial_t h = 0, \quad \forall\, t > 0. \tag{7}$$

Consequently, for any classical solution (u, h, p) to this system, the right-hand side of (3) must have zero mean:

$$\int_0^L \phi(u, p, h) = 0.$$

It can be achieved thanks to a good choice of the constant normalizing the pressure which is consequently uniquely defined. The pressure can be then decomposed, for instance, as

$$p = p_0 + c, \tag{8}$$

where one chooses to impose

$$\int_{\mathcal{F}(t)} p_0 = 0, \tag{9}$$

and where c satisfies

$$c(t) = \frac{1}{L}\int_0^L e_2 \cdot (\sigma(u, p_0))(x, h(x, t), t)(-\partial_x h(x, t)\, e_1 + e_2)\, \mathrm{d}x. \tag{10}$$

This constant c is the Lagrange multiplier associated with the constraint (7). Another mathematical way to compel the solution with this compatibility condition, without defining the physical average of the fluid pressure c, is to introduce the L^2-projection operator on the set of L-periodic functions whose averages are equal to zero on $(0, L)$, denoted M_s, and rewrite (3) as

$$\rho_s \partial_{tt} h - \beta \partial_{xx} h + \alpha \partial_{xxxx} h - \gamma \partial_{txx} h = M_s \phi(u, p, h). \tag{11}$$

This is the choice made in [39].

2.1. Main Result

In what follows, we call (BF) the "beam-fluid" system (1)–(2)–(3)–(4)–(5)–(6)–(8)–(9)–(10). We study herein the (BF) system, completed with initial conditions:

$$h(x, 0) = h^0(x), \quad x \in (0, L), \tag{12}$$

$$\partial_t h(x, 0) = \dot{h}^0(x), \quad x \in (0, L), \tag{13}$$

$$u(x, y, 0) = u^0(x, y), \quad (x, y) \in \{x \in (0, L), y \in (0, h_0(x))\} =: \mathcal{F}^0. \tag{14}$$

Our aim is to study this Cauchy problem and, specifically, to prove the existence of a unique global-in-time strong solution.

We first give some notations and definitions and make precise the functional framework. For any given positive function $b \in C_\sharp(0, L)$, that is the set of continuous and L-periodic functions on $\mathbb{R}$, we define

$$\Omega_b := \{(x, y) \in \mathbb{R}^2 \text{ s. t. } x \in (0, L), y \in (0, b(x))\}.$$

With this definition $\mathcal{F}(t) = \Omega_{h(t,\cdot)}$. We denote by $C^\infty_\sharp(\Omega_b)$ the restriction on Ω_b of L-periodic functions in x indefinitely differentiable on

$$\mathcal{O}_b = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } y \in (0, b(x))\}$$

Note that $\mathcal{O}_b + Le_1 \subset \mathcal{O}_b$ and $\mathcal{O}_b = \cup_{k\in\mathbb{Z}} \Omega_b + Lke_1$. We introduce the classical spaces $L^p_\sharp(\Omega_b)$ and $H^m_\sharp(\Omega_b)$ respectively as the closures of $C^\infty_\sharp(\Omega_b)$ in $L^p(\Omega_b)$ or $H^m(\Omega_b)$. We define in the same way $C^s_\sharp(0, L)$, $L^p_\sharp(0, L)$ and $H^m_\sharp(0, L)$. More generally, the subscript $\sharp$ stands for the periodic version in the first variable of a function space. We emphasize that, contrary to the usual convention, we consider that time is the last variable of a function. This enables to write a unified definition for periodic functions whether they depend on one space variable only (such as the height function h) or two space variables (such as the velocity-field u). Finally, we set:

$$L^2_{\sharp,0}(0, L) := \left\{ f \in L^2_\sharp(0, L) \text{ s.t. } \int_0^L f = 0 \right\},$$

and, in the same way,

$$L^2_{\sharp,0}(\Omega_b) := \left\{ f \in L^2_\sharp(\Omega_b) \text{ s.t. } \int_{\Omega_b} f = 0 \right\}.$$

Then the projection operator M_s, that is applied to Equation (3) leading to Equation (11), is the orthogonal-projector from $L^2_\sharp(0, L)$ onto $L^2_{\sharp,0}(0, L)$.

We state our main result as follows:

Theorem 1. *Let us consider* $\alpha > 0$, $\beta \geqq 0$ *and* $\gamma > 0$. *Assume that the initial data* $(h^0, \dot{h}^0, u^0)$ *satisfy:*

- $(h^0, \dot{h}^0) \in H^3_\sharp(0, L) \times H^1_\sharp(0, L)$,
- $u^0 \in H^1_\sharp(\mathcal{F}^0)$,
- *no-slip condition is fulfilled initially:*

$$u^0(x, 0) = 0, \quad u^0(x, h^0(x)) = \dot{h}^0(x)e_2, \quad \forall\, x \in (0, L), \tag{15}$$

- *no-contact and incompressibility compatibility conditions are fulfilled initially:*

$$\min_{x\in[0,L]} h^0(x) > 0, \quad \int_0^L \dot{h}^0 = 0, \tag{16}$$

$$\mathrm{div} u^0 = 0 \quad \text{on } \mathcal{F}^0. \tag{17}$$

Then (BF) has a unique global-in-time strong solution.

A precise definition of what is a "strong solution" is given in Section 3. Our proof for Theorem 1 follows a classical scheme: local-in-time existence and uniqueness of solutions, blow-up alternative and a priori estimates. Local-in-time existence and uniqueness of strong solutions has already been tackled in [39] for clamped boundary conditions instead of periodic boundary conditions. In Section 3, we explain shortly how this result can be adapted to our functional framework with periodic lateral boundary conditions. This construction leads to the existence of a unique maximal solution for any given initial data, that blows up in finite time if and only if the quantity

$$\mathcal{C}(t) := \sup_{x\in[0,L]} \frac{1}{h(x,t)} + \int_0^L \left(\alpha|\partial_{xxx}h(x,t)|^2 + \gamma|\partial_{tx}h(x,t)|^2\right) \mathrm{d}x \\ + \int_0^L \int_0^{h(x,t)} \mu|\nabla u(x, y, t)|^2 \mathrm{d}x\mathrm{d}y \tag{18}$$

diverges in finite time (see Corollary 1). Note that existence of weak solutions as long as the beam does not touch the bottom of the fluid cavity can be obtained also by adapting [5,23] or [41] to our setting.

In this paper, the main novelty is the computation, for any local-in-time strong solutions to (BF), of a new a priori estimate on $\mathcal{C}$, defined by (18). This estimate enables us to derive a regularity estimate valid on any given time interval $(0, T)$. We emphasize that, in order to obtain these estimates, we have to assume that $\alpha > 0$ and $\gamma > 0$. This ensures first that the elastic boundary remains regular, second that the beam dissipates energy. Whether or not these assumptions can be dropped remains an open question (notice that existence of weak solutions before contact is still valid for $\alpha = \gamma = 0$ and $\beta > 0$).

2.2. Formal Argument

Before considering the full Navier–Stokes/beam system, and in order to illustrate the different steps of the proof, let us first consider a reduced model for which we derive similar bounds (at a formal level for conciseness). This coupled system can be written as

$$\rho_s \partial_{tt} b - \beta \partial_{xx} b + \alpha\, \partial_{xxxx} b - \gamma\, \partial_{xxt} b = q, \qquad \text{on } (0, L), \tag{19}$$

$$\partial_t b = \partial_x [b^3 \partial_x q], \qquad \text{on } (0, L), \tag{20}$$

where b stands for the deformation of the beam (analogue of h) and q denotes the fluid pressure. To complete these equations, we require that q satisfies

$$\int_0^L q = 0, \quad \forall t > 0, \tag{21}$$

and we impose periodic boundary conditions in x. We note that this system is formally related to (BF) in the regime $h << 1$. Indeed, in this regime, the aspect ratio of the fluid film is very small. It is then known that the fluid pressure is computed by integrating a Reynolds equation and that the action of the fluid film on the beam is mostly due to pressure terms. The above toy-model is thus constructed by coupling the Reynolds equation (20) satisfied by the fluid pressure with the beam equation (19) in which only pressure terms are kept. We refer to [36, Section 5.B] for a detailed derivation of the Reynolds equation.

Let (b, q) be a (classical) L-periodic solution to (19)–(21) on $(0, T)$ with $T > 0$. First, multiplying (19) by $\partial_t b$ and combining with (20) multiplied by q yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L \left(\rho_s|\partial_t b|^2 + \beta|\partial_x b|^2 + \alpha|\partial_{xx} b|^2\right)\right] + \int_0^L \left(\gamma|\partial_{tx} b|^2 + b^3|\partial_x q|^2\right) = 0.$$

We obtain that there exists a constant C_0 depending only on initial data for which:

$$\sup_{t\in(0,T)} \left(\rho_s\|\partial_t b\|^2_{L^2_\sharp(0,L)} + \alpha\|b\|^2_{H^2_\sharp(0,L)} + \beta\|b\|^2_{H^1_\sharp(0,L)}\right) + \int_0^T \left(\gamma\|\partial_t b\|^2_{H^1_\sharp(0,L)} + \|b^{\frac{3}{2}}\partial_x q\|^2_{L^2_\sharp(0,L)}\right) \leqq C_0. \tag{22}$$

In what follows, C_0 denotes a constant depending only on the initial data but which may vary between lines. To obtain a lower bound on b, we multiply (19) by $-\partial_{xx} b$ and integrate over $(0, L)$. We then integrate by parts in space. By taking into account the periodic boundary conditions and (20), we obtain:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L \left(\frac{\gamma}{2}|\partial_{xx} b|^2 - \rho_s \partial_t b\, \partial_{xx} b\right)\right] + \int_0^L \left(\beta|\partial_{xx} b|^2 + \alpha|\partial_{xxx} b|^2 - \rho_s|\partial_{tx} b|^2\right) \\
&= -\int_0^L q\partial_{xx} b = \int_0^L \partial_x q \partial_x b = \int_0^L b^3 \partial_x q \left(\frac{1}{b^3}\partial_x b\right) \\
&= -\frac{1}{2}\int_0^L b^3 \partial_x q\ \partial_x\left[\frac{1}{b^2}\right] = \frac{1}{2}\int_0^L \frac{\partial_t b}{b^2}.
\end{aligned}$$

Finally, we deduce:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L \frac{1}{2}\left(\gamma|\partial_{xx}b|^2+\frac{1}{b}\right)-\int_0^L \rho_s\partial_t b\,\partial_{xx}b\right] + \int_0^L\left(\beta|\partial_{xx}b|^2+\alpha|\partial_{xxx}b|^2\right)=\int_0^L \rho_s|\partial_{tx}b|^2. \tag{23}$$

Combining with the previous bound (22), we get that there exists a constant C_0 such that:

$$\sup_{t\in(0,T)}\left(\gamma\|b\|^2_{H^2_\sharp(0,L)}+\|b^{-1}\|_{L^1_\sharp(0,L)}\right) + \int_0^T \alpha\|b\|_{H^3_\sharp(0,L)}+\beta\|b\|^2_{H^2_\sharp(0,L)}\leqq C_0. \tag{24}$$

Note that, if $\rho_s\neq 0$, to obtain the previous estimate one should compute an upper bound for

$$\int_0^T\int_0^L \rho_s|\partial_{tx}b|^2,\qquad \sup_{t\in(0,T)}\int_0^L \rho_s\partial_t b\,\partial_{xx}b.$$

From (22) these terms are bounded if $\gamma>0$ and if γ or α is strictly positive respectively. At this point, we call a real-analysis lemma which states that there exists a continuous function $D_{\min}$ for which:

$$\|b^{-1}\|_{L^\infty_\sharp(0,L)}\leqq D_{\min}(\|b\|_{H^2_\sharp(0,L)},\|b^{-1}\|_{L^1_\sharp(0,L)})$$

(see "Appendix A.1" for a proof). The first consequence of this inequality is that (24) implies that b remains away from zero uniformly in time. Combining this information with the dissipation estimate (22) we obtain that there exists a constant C_0 for which:

$$\sup_{t\in(0,T)}\left(\|b^{-1}(\cdot,t)\|_{L^\infty_\sharp(0,L)}\right)+\int_0^T\|q\|^2_{H^1_\sharp(0,L)}\leqq C_0.$$

Consequently the pressure is bounded in $L^2(0,T;H^1_\sharp(0,L))$. Next we derive a regularity estimate for the deformation b by multiplying (19) by $-\partial_{txx}b$. This yields after integration by parts in space:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L \rho_s|\partial_{tx}b|^2+\alpha|\partial_{xxx}b|^2+\beta|\partial_{xx}b|^2\right]+\gamma\int_0^L|\partial_{txx}b|^2=\int_0^L q\,\partial_{txx}b.$$

Thanks to the $L^2(0,T;H^1_\sharp(0,L))$-bound on q, we reach the required estimate that enables us to extend solutions globally in time:

$$\sup_{t\in(0,T)}\left(\alpha\|b\|^2_{H^3_\sharp(0,L)}+\beta\|b\|^2_{H^2_\sharp(0,L)}+\rho_s\|\partial_t b\|^2_{H^1_\sharp(0,L)}\right) +\gamma\int_0^T\|\partial_t b\|^2_{H^2_\sharp(0,L)}\leqq C_0. \tag{25}$$

This ends the formal proof of a no collision result, on the one hand, and of a global-in-time existence of strong solutions, on the other hand.

In Section 4, we prove that comparable estimates hold true for the complete coupled system (BF). First, considering the full beam/Navier–Stokes system, the analogue of (22) corresponds to the (already-known) classical decay of kinetic energy. To obtain a similar estimate to (24), we multiply (11) by $-\partial_{xx}h$ and multiply (1) by a suitable extension of $-\partial_{xx}h$. The choice of this extension is a key point of the proof (see Section 4.2). We then obtain an identity similar to (23) with additional remainder terms that we bound thanks to the energy estimate and for which a control of h in $L^2(0, T; H^3_\sharp(0, L)) \cap L^\infty(0, T; H^2_\sharp(0, L))$, resp. a control of $\partial_t h$ in $L^2(0, T; H^1_\sharp(0, L))$ is needed (hence $\alpha > 0$, resp. $\gamma > 0$). The extension of the last estimate (25) is more involved. Indeed, when dealing with the full Navier–Stokes/beam system, we also have to control the fluid velocity-field u in $L^\infty(0, T; H^1_\sharp(\mathcal{F}(t)))$ (and not only the pressure field as for the toy model). When working in cylindrical domains, such an estimate is obtained by multiplying (1) with $\partial_t u$ and by applying elliptic estimates for the Stokes system in order to bound the convective terms. However, these elliptic estimates are classically proven in $C^{1,1}$-domains or $W^{2,\infty}$-domains [2,15]. Here as h is, at this stage of the proof, merely $L^\infty(0, T; H^2_\sharp(0, L))$, we cannot directly apply these standard regularity properties, so we need to extend the elliptic results for the Stokes system to domains which are only subgraphs of H^2-functions and analyze precisely the dependency of the associated elliptic estimates with respect to the norms of h (see Lemma 1). This proof is an adaptation to a periodic framework of a lemma that can be found in [22]. Moreover, as the fluid domain is moving with time, instead of $\partial_t u$, we need to consider a multiplier that takes into account this motion. The most natural choice is $\partial_t u + u \cdot \nabla u$, but this function is not divergence-free and consequently pressure terms appear that cannot be handled easily. To avoid this difficulty, we mimic the method used in [9] in the framework of fluid/solid interactions. We introduce a divergence-free multiplier avoiding the introduction of the pressure in the regularity estimate. Moreorever this multiplier is chosen so that the associated multiplier for the structure equation is $\partial_{tt}h$. Nevertheless, special attention needs to be paid since the structure motion is, once again, less regular than when considering fluid/solid interactions.

The outline of this paper is as follows. In next section, we focus on the change of variables turning the beam/fluid system into a quasilinear system in a fixed geometry. We recall the construction of local-in-time strong solutions of [39] and adapt this result to our periodic boundary conditions framework. We end the section by a technical proposition dealing with elliptic estimates for the inhomogeneous Stokes system in H^2-subgraph domains. The third and last sections are devoted to the proof of Theorem 1. They are divided into three subsections corresponding respectively to the extension of the three estimates (22), (24) and (25) to solutions of the coupled problem (BF).

3. Local-in-Time Strong Solutions and Technical Lemmas

In this section, we first adapt the construction of local-in-time strong solutions of [39] to our periodic setting. To this end, we will apply the following change of variables:

$$\hat{f}(x,z) = f(x, h(x)z), \quad \forall (x,z) \in \Omega_1. \tag{26}$$

To measure the regularity of such a change of variable, the following technical proposition is required:

Proposition 1. *Let us consider* $h \in H^2_\sharp(0,L)$ *satisfying* $\min_{x\in[0,L]} h(x) > 0$. *Then for any given* $m \leqq 2$,

- *the mapping* $f \mapsto \hat{f}$ *defined by* (26) *realizes a linear homeomorphism from* $H^m_\sharp(\Omega_h)$ *onto* $H^m_\sharp(\Omega_1)$,
- *there exists a non decreasing function* $K^e_m : [0,+\infty) \to (0,\infty)$ *such that, if we assume moreover that* $\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} \leqq R_0$ *then there holds:*

$$\|\hat{f}\|_{H^m_\sharp(\Omega_1)} \leqq K^e_m(R_0)\|f\|_{H^m_\sharp(\Omega_h)},$$
$$\|f\|_{H^m_\sharp(\Omega_h)} \leqq K^e_m(R_0)\|\hat{f}\|_{H^m_\sharp(\Omega_1)}.$$

Proof. The proof is standard. For $m \in \{0,1\}$ the result easily derives from the fact that $h \in W^{1,\infty}(0,L)$ is bounded from below by a strictly positive constant. For $m = 2$, the key point is that the motion of the upper boundary is tranverse only so that we combine the regularity of h with the following tensorization of the space $H^1_\sharp(\Omega_1)$:

$$H^1_\sharp(\Omega_1) = H^1_\sharp((0,L)\times(0,1)) = L^2_\sharp(0,L; H^1(0,1)) \cap H^1_\sharp(0,L; L^2(0,1)).$$

The most delicate point enters the computation of $\|\partial_{xx}\hat{f}\|_{L^2_\sharp(\Omega_1)}$. We have:

$$\partial_{xx}\hat{f} = \widehat{\partial_{xx} f} + 2h'z\widehat{\partial_{xy} f} + h''z\widehat{\partial_y f} + (h'z)^2\widehat{\partial_{yy} f},$$

in which the worst term is $h''z\widehat{\partial_y f}$. It is bounded in $L^2_\sharp(\Omega_1)$ since $h'' \in L^2_\sharp(0,L)$ and

$$\widehat{\partial_y f} \in H^1_\sharp(\Omega_1) \hookrightarrow L^\infty_\sharp(0,L; L^2(0,1)) \cap L^2_\sharp(0,L; L^\infty(0,1)).$$

□

3.1. Construction of Local-in-Time Solutions

As explained previously, the local-in-time existence and uniqueness of strong solutions to (BF) are tackled in [39] with no normalizing condition for the pressure ((8)–(9)–(10) is replaced with (11)), with homogeneous Dirichlet boundary conditions for the fluid velocity on the part of the boundary that is not elastic and with "clamped" boundary conditions for the structure. Namely, instead of periodic boundary conditions, the displacement $\eta = h - 1$ satisfies

$$\eta(0,t) = \eta(L,t) = \partial_x\eta(0,t) = \partial_x\eta(L,t) = 0, \quad \forall t \in (0,T).$$

The proof of the existence of solutions follows a classical method, also introduced in [26,48] when dealing with fluid/solid interactions. To look for solutions on a time-interval $(0, T)$, new unknowns $(\hat{u}, \hat{p})$ are first introduced applying the transformation (26):

$$u(x, y, t) = \hat{u}\left(x, \frac{y}{h(x,t)}, t\right), \quad p(x, y, t) = \hat{p}\left(x, \frac{y}{h(x,t)}, t\right), \quad (x, y) \in \mathcal{F}(t). \tag{27}$$

These new velocity-field and pressure $(\hat{u}, \hat{p})$ are defined in the cylindrical domain $\Omega_1 \times (0, T)$ and (u, p, h) is a solution to (1)–(2)–(3)–(4)–(5)–(6)–(11) if and only if the triplet $(\hat{u}, \hat{p}, h)$ is solution to a coupled system of quasilinear PDEs that we choose not to write here for the sake of conciseness. The core of the existence and uniqueness result is the study of this nonlinear system. First, the author analyzes, via a semi-group approach, the resolution of the linear system obtained by linearizing around $\eta = 0$ (or $h = 1$), $\hat{u} = 0$, $\hat{p} = 0$. This study is based on an accurate treatment of the added mass effect of the fluid on the structure through an appropriate splitting of the fluid load. Then, the nonlinear terms are estimated and the author proves that they remain small for a small time. The local-in-time existence and uniqueness of a solution to the system of nonlinear PDEs is finally obtained by a standard fixed point argument.

In our periodic framework, computation of nonlinearities might be reproduced without change while the semi-group approach might be adapted in the spirit of [38]. Consequently, for any initial data such that

$$h^0 \in H^3_\sharp(0, L), \quad \dot{h}^0 \in H^1_\sharp(0, L), \quad u^0 \in H^1_\sharp(\mathcal{F}^0), \tag{28}$$

and satisfying the compatibility conditions:

$$\min_{x \in [0,L]} h^0(x) > 0, \qquad \int_0^L \dot{h}^0 = 0, \tag{29}$$

$$\operatorname{div} u^0 = 0, \quad \text{on } \mathcal{F}^0, \tag{30}$$

$$u^0(x, h^0(x)) = \dot{h}^0(x) e_2, \quad x \in (0, L), \tag{31}$$

$$u^0(x, 0) = 0, \qquad x \in (0, L), \tag{32}$$

we obtain local-in-time existence and uniqueness of a strong solution $(\hat{u}, \hat{p}, h)$ to the Cauchy problem associated with the translation of (1)–(2)–(3)–(4)–(5)–(6)–(11) in a fixed geometry, completed with periodic boundary conditions. The solution verifies:

$$\hat{u} \in H^1(0, T; L^2_\sharp(\Omega_1)) \cap C([0, T]; H^1_\sharp(\Omega_1)) \cap L^2(0, T; H^2_\sharp(\Omega_1)), \tag{33}$$

$$\hat{p} \in L^2(0, T; H^1_\sharp(\Omega_1)), \tag{34}$$

$$h \in H^2(0, T; L^2_\sharp(0, L)) \cap L^2(0, T; H^4_\sharp(0, L)) \tag{35}$$

$$h^{-1} \in L^\infty((0, L) \times (0, T)). \tag{36}$$

We emphasize that, following the proof of [39], the pressure $\hat{p}$ is defined up to a constant for now. The regularity statement (35) and (36) ensures that the function h is Lipschitz on $(0, L) \times [0, T]$. Moreover, since h satisfies also

$$\frac{1}{h} \in W^{1,\infty}((0,L)\times(0,T)), \tag{37}$$

we obtain that the domain $\mathcal{F}(t)$ and the non cylindrical domain defined by:

$$\mathcal{Q}_t := \{(x,y,s), \quad x\in(0,L), \quad s\in(0,t), \quad y\in(0,h(s,x))\}, \quad \forall t \leqq T,$$

are both Lipschitz open subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Going back to the moving domain by inverting the transformation (27), we define strong solutions of (BF) as follows

Definition 1. Let the initial data $(h^0, \dot{h}^0, u^0) \in H^3_\sharp(0,L)\times H^1_\sharp(0,L)\times H^1_\sharp(\mathcal{F}^0)$ satisfy (29)–(30)–(31)–(32) and let $T>0$. A strong solution to (BF) on $(0,T)$, associated with the initial data $(h^0,\dot{h}^0,u^0)$, is a quadruplet (h,u,p,c) satisfying:

- h, u, p and c have the following regularity:

$$h \in H^2(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^4_\sharp(0,L)), \tag{38}$$

$$h^{-1} \in L^\infty((0,L)\times(0,T)), \tag{39}$$

$$u\in H^1_\sharp(\mathcal{Q}_T), \quad \nabla^2 u \in L^2_\sharp(\mathcal{Q}_T), \quad c\in L^2(0,T), \quad \nabla p \in L^2_\sharp(\mathcal{Q}_T), \tag{40}$$

- Equations (1)–(2) are satisfied almost everywhere in $\mathcal{Q}_T$,
- Equations (8)–(9)–(10) are satisfied almost everywhere in $(0,T)$,
- Equations (3)–(5)–(6) are satisfied almost everywhere in $(0,L)\times(0,T)$,
- Equations (12)–(13)–(14) are satisfied almost everywhere in $(0,L)$ and $\mathcal{F}^0$.

We emphasize that the pressure p in our definition is completely fixed and that u is a time–space function. Hence, the condition $u\in H^1_\sharp(\mathcal{Q}_T)$ involves both time and space derivatives of u, whereas ∇p involves space derivatives only. The construction that we describe above, adapted from [39], yields the following existence and uniqueness theorem:

Theorem 2. *Let us consider $\alpha>0$, $\beta \geqq 0$ and $\gamma>0$. Assume that the initial data $(h^0,\dot{h}^0,u^0)$ belong to $H^3_\sharp(0,L)\times H^1_\sharp(0,L)\times H^1_\sharp(\mathcal{F}^0)$ and satisfy the compatibility conditions (29)–(30)–(31)–(32). There exists $T_0>0$ such that for any $0<T<T_0$, there exists a unique strong solution to (BF) on $(0,T)$.*

Proof. In the whole proof initial data $(h^0,\dot{h}^0,u^0)$ are fixed. The only points that we want to make clear here are:

- the link between the regularity (33)–(34)–(35)–(36) of the solution $(\hat{u},\hat{p},h)$ to the nonlinear system in a fixed geometry and the regularity statements of Definition 1;
- the computation of the Lagrange multiplier c that we introduce here and that does not appear in [38,39].

Let $(\hat{u},\hat{p},h)$ be the solution to (1)–(2)–(3)–(4)–(5)–(6)–(11) written in a fixed geometry, that one constructs adapting the arguments of [38,39] as explained in introduction. The deformation $h(\cdot,t)$ and its inverse $1/h(\cdot,t)$ are then uniformly (with respect to $t\in[0,T]$) bounded in $H^2_\sharp(0,L)$ and $L^\infty_\sharp(0,L)$. So, we construct

(u, p) via (26) and apply Proposition 1 to obtain a fluid velocity and a fluid pressure that satisfy their contribution to (40). Noting that, in the aforementioned construction, p is defined up to a time-dependent constant and that $\phi(u, p+c, h) = \phi(u, p, h) + c$, we fix p by requiring further that:

$$\int_0^L \phi(u, p, h) = 0, \quad \forall t \in (0, T). \tag{41}$$

We then write $p = p_0 + c$ with p_0 satisfying (9) and with c being fixed by (10). Due to the regularity of h, u and p_0 we obtain finally that c belongs to $L^2(0, T)$.

Conversely, for any given strong solution (h, u, p, c) of (BF) in the sense of Definition 1, we construct $(\hat{u}, \hat{p})$ by (27) and refer to Proposition 1 again yielding:

$$\hat{u} \in H^1(0, T; L^2_\sharp(\Omega_1)) \cap L^2(0, T; H^2_\sharp(\Omega_1)), \quad \hat{p} \in L^2(0, T; H^1_\sharp(\Omega_1)).$$

We then apply [40, Theorem 3.1] and deduce that $\hat{u} \in C([0, T]; H^1_\sharp(\Omega_1))$ and get that, for T small enough, $(\hat{u}, \hat{p}, h)$ is the unique solution to (1)–(2)–(3)–(4)–(5)–(6)–(11) written in a fixed geometry, as constructed by adapting the arguments of [38,39]. □

Remark 1. From the regularity we just derived for $\hat{u}$, we deduce that, for any strong solution (u, p, h) the mapping $t \mapsto \int_{\mathcal{F}(t)} |\nabla u|^2(t)$ belongs to $C^0([0, T])$.

Finally, we obtain that (BF) is wellposed locally in time. Following [39], it appears that we might choose the time T_0 in Theorem 2 to be fixed by

$$\|h^0\|_{H^3_\sharp(0,L)} + \|\dot{h}^0\|_{H^1_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} + \|u^0\|_{H^1_\sharp(\mathcal{F}^0)}$$

only (see the computation of T_0 at item (i), page 408). Then the following blow-up alternative can be classically stated:

Corollary 1. *Let $\alpha > 0$, $\beta \geqq 0$ and $\gamma > 0$ be given. Assume that the initial data $(h^0, \dot{h}^0, u^0)$ belong to $H^3_\sharp(0, L) \times H^1_\sharp(0, L) \times H^1_\sharp(\mathcal{F}^0)$ and satisfy the compatibility conditions (29)–(30)–(31)–(32). Then (BF) completed with initial conditions (12)–(14) has a unique non-extendable strong solution $(T^*, (h, u, p, c))$. Furthermore, we have the following alternative:*

(i) *either $T^* = +\infty$*
(ii) *either $T^* < \infty$ and*

$$\limsup_{t \to T^*} \Big(\|h(\cdot, t)\|_{H^3_\sharp(0,L)} + \|\partial_t h(\cdot, t)\|_{H^1_\sharp(0,L)} + \|h^{-1}(\cdot, t)\|_{L^\infty_\sharp(0,L)} + \|u(\cdot, t)\|_{H^1_\sharp(\mathcal{F}(t))} \Big) = +\infty.$$

The aim of Section 4 is to prove that the second alternative (ii) never holds and consequently that the solution is defined on any finite time interval $(0, T)$. However, before going any further we focus on the elliptic regularity properties of the inhomogenous Stokes system in a subgraph domain.

3.2. Elliptic Estimate

In this subsection we derive elliptic estimates for the inhomogeneous Stokes problem in a domain Ω_h with $h \in H^2_\sharp(0,L)$ such that $h^{-1} \in L^\infty_\sharp(0,L)$. With this regularity, the domain is neither $C^{1,1}$ nor $W^{2,\infty}$ and one cannot apply standard elliptic regularity results. Nevertheless, we take advantage here of the fact that Ω_h is a subgraph so that the change of variable transforming Ω_h into a flat domain (namely Ω_1) can be chosen to be smooth in the transverse variable (see χ_h below). This remark enables us to extend the classical method with h belonging merely to $H^2_\sharp(0,L)$. Such an estimate is a key argument in the derivation of the regularity estimates for the solution of the nonlinear system (BF).

For simplicity, we fix $\mu = 1$ in this part. Let us consider source terms and a boundary condition

$$(f,g) \in L^2_\sharp(\Omega_h) \times H^1_\sharp(\Omega_h), \quad \dot\eta \in H^{\frac{3}{2}}_\sharp(0,L).$$

We aim at studying the regularity properties of L-periodic (with respect to x) solutions to

$$-\Delta u + \nabla p_0 = f, \quad \text{in } \Omega_h, \tag{42}$$

$$\operatorname{div} u = g, \quad \text{in } \Omega_h, \tag{43}$$

completed with boundary conditions:

$$u(x,h(x)) = \dot\eta(x) e_2, \qquad \forall x \in (0,L), \tag{44}$$

$$u(x,0) = 0, \qquad \forall x \in (0,L). \tag{45}$$

Integrating $\operatorname{div} u = g$ over Ω_h implies that the boundary velocity $\dot\eta$ has to satisfy

$$\int_0^L \dot\eta = \int_{\Omega_h} g. \tag{46}$$

The left-hand side of (46) does not involve the deformation h, because the deformation as well as the boundary velocity are vertical. In what follows, we restrict to data $g \in L^2_{\sharp,0}(\Omega_h)$ and $\dot\eta \in L^2_{\sharp,0}(0,L)$ for which (46) is clearly satisfied.

Remark 2. Note that, for this inhomogeneous Stokes problem, with Dirichlet boundary conditions, the pressure p_0 is defined up to a constant. Consequently, we enforce the uniqueness of the pressure by imposing:

$$\int_{\Omega_h} p_0 = 0.$$

The main result of this section is:

Lemma 1. *For any $h \in H^2_\sharp(0,L)$ such that $h^{-1} \in L^\infty_\sharp(0,L)$, source terms and boundary condition*

$$(f,g) \in L^2_\sharp(\Omega_h) \times (H^1_\sharp(\Omega_h) \cap L^2_{\sharp,0}(\Omega_h)), \quad \dot\eta \in H^{\frac{3}{2}}_\sharp(0,L) \cap L^2_{\sharp,0}(0,L),$$

there exists a unique solution $(u, p_0) \in H^2_\sharp(\Omega_h) \times (H^1_\sharp(\Omega_h) \cap L^2_{\sharp,0}(\Omega_h))$ *to the Stokes system* (42)–(43)–(44)–(45)*. Moreover, there exists a non-decreasing function* $K^s : [0, \infty) \to (0, \infty)$ *such that, if we assume* $\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} \leqq R_0$ *then, this solution satisfies:*

$$\|u\|_{H^2_\sharp(\Omega_h)} + \|p_0\|_{H^1_\sharp(\Omega_h)} \leqq K^s(R_0)\left(\|f\|_{L^2_\sharp(\Omega_h)} + \|g\|_{H^1_\sharp(\Omega_h)} + \|\dot{\eta}\|_{H^{\frac{3}{2}}_\sharp(0,L)} \right). \tag{47}$$

The remainder of this section is devoted to the proof of Lemma 1. This proof is an adaptation, to our periodic framework, of the computations on the Stokes problem that can be found in [22], which, in itself, uses ideas from [49]. Compared to [22], we also carefully analyze the dependance of the constant K^s on h in inequality (47). To obtain the expected dependency, we assume throughout this section that:

$$\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} \leqq R_0,$$

and show that K^s depends only on R_0.

First step: Rewriting of the Stokes system in a given geometry. As in [22] we compute regularity estimates for solutions to (42)–(45) by studying the Stokes system transported in a geometry which does not depend on the deformation h. Namely, we derive regularity estimates on $(\hat{u}, \hat{p})$ defined by (27). Indeed, thanks to Proposition 1, we remark that $(u, p) \in H^2_\sharp(\Omega_h) \times H^1_\sharp(\Omega_h)$ is a solution to (42)–(45) if and only if $(\hat{u}, \hat{p}) \in H^2_\sharp(\Omega_1) \times H^1_\sharp(\Omega_1)$ is a solution to the following Stokes-like system

$$-\operatorname{div}[(A_h \nabla)\hat{u}] + (B_h \nabla)\hat{p}_0 = \tilde{f}, \quad \text{in } \Omega_1, \tag{48}$$

$$\operatorname{div}(B_h^\top \hat{u}) = \tilde{g}, \quad \text{in } \Omega_1, \tag{49}$$

completed with boundary conditions:

$$\hat{u}(x, 1) = \dot{\eta}(x) e_2, \qquad \forall x \in (0, L), \tag{50}$$

$$\hat{u}(x, 0) = 0, \qquad \forall x \in (0, L), \tag{51}$$

where (A_h, B_h) and $(\tilde{f}, \tilde{g})$ are explicit. Indeed, by introducing the mapping $\chi_h(x, z) = (x, h(x)z)$, for $(x, z) \in \Omega_1$, we obtain:

$$B_h := \operatorname{cof} \nabla \chi_h = \begin{pmatrix} h & -h'z \\ 0 & 1 \end{pmatrix}, \tag{52}$$

$$A_h := \frac{1}{h}(\operatorname{cof} \nabla \chi_h)^{\mathrm{T}} \operatorname{cof} \nabla \chi_h = \begin{pmatrix} h & -h'z \\ -h'z & \frac{1}{h} + \frac{(h'z)^2}{h} \end{pmatrix}, \tag{53}$$

and the transported source terms:

$$\tilde{f} := h\hat{f}, \quad \tilde{g} := h\hat{g}.$$

We note that $\tilde{f} \in L^2_\sharp(\Omega_1)$, $\tilde{g} \in L^2_{\sharp,0}(\Omega_1) \cap H^1_\sharp(\Omega_1)$. Thus, thanks to Proposition 1, to prove Lemma 1 it is sufficient to derive similar estimates but on the transported unknowns $(\hat{u}, \hat{p})$ solution of (48)–(51). Namely, we obtain that there exists a unique $(\hat{u}, \hat{p}_0) \in H^2_\sharp(\Omega_1) \times (H^1_\sharp(\Omega_1) \cap L^2_{\sharp,0}(\Omega_1))$ solution of (48)–(51) that satisfies

$$\|\hat{u}\|_{H^2_\sharp(\Omega_1)} + \|\hat{p}_0\|_{H^1_\sharp(\Omega_1)} \leqq K\left(\|\tilde{f}\|_{L^2_\sharp(\Omega_1)} + \|\tilde{g}\|_{H^1_\sharp(\Omega_1)} + \|\dot{\eta}\|_{H^{\frac{3}{2}}_\sharp(0,L)} \right), \quad (54)$$

where the constant K depends only on R_0. Since the matrices B_h and A_h are in $H^1_\sharp((0, L); H^s(0, 1))$, for any $s \geqq 0$, we have that A_h and B_h belong to a multiplier space of $H^1(\Omega_1)$. We refer the reader to [22, Lemma 6] for more details. In particular, we obtain that for any $v \in H^2_\sharp(\Omega_1)$, there holds $\mathrm{div}[(A_h\nabla)v] \in L^2_\sharp(\Omega_1)$ and, for any $q \in H^1_\sharp(\Omega_1)$, there holds $(B_h\nabla)q \in L^2_\sharp(\Omega_1)$. Thanks to Piola identity, we also have:

$$\mathrm{div}(B_h^\top v) = B_h^\top : \nabla v, \quad (55)$$

so that, for any $v \in H^2_\sharp(\Omega_1)$, there holds $\mathrm{div}(B_h^\top v) \in H^1_\sharp(\Omega_1)$. Consequently the assumptions on the deformation h are compatible with the expected regularity on $(\hat{u}, \hat{p})$.

Remark 3. Let us mention that we get estimates for a pressure $\hat{p}_0$ such that

$$\int_{\Omega_1} \hat{p}_0 = 0.$$

Through the change of variables (26), this implies that the pressure q, defined by $q(x, y) = \hat{p}_0(x, y/h(x))$ and on which we deduce an estimate, verifies the following constraint

$$\int_{\Omega_h} \frac{q}{h} = 0.$$

Thus, the pressure we compute with this method does not match the one mentioned in Lemma 1. Nevertheless, the effective pressure p_0 mentioned in Lemma 1 reads $P_h q$ where P_h stands for the L^2-orthogonal projector on $L^2_{\sharp,0}(\Omega_h)$:

$$p_0 = P_h q := q - \frac{1}{|\Omega_h|}\int_{\Omega_h} q$$

satisfying $\|p_0\|_{H^1_\sharp(\Omega_h)} \leqq \|q\|_{H^1_\sharp(\Omega_h)}$. Hence, we prove (47) with p_0 replaced by q.

Let us now study precisely the existence and uniqueness of $(\hat{u}, \hat{p})$ in $H^1_\sharp(\Omega_1) \times L^2_{\sharp,0}(\Omega_1)$ and derive elliptic estimates in $H^2_\sharp(\Omega_1) \times H^1_\sharp(\Omega_1)$.

Second step: Lifting of the Dirichlet boundary conditions. As $\dot{\eta} \in H^{\frac{3}{2}}_\sharp(0, L)$ there exists $u_{\dot{\eta}} \in H^2_\sharp(\Omega_1)$ such that $u_{\dot{\eta}|_{z=1}} = \dot{\eta}e_2$, $u_{\dot{\eta}|_{z=0}} = 0$ and

$$\|u_{\dot\eta}\|_{H^2_\sharp(\Omega_1)} \leqq C\|\dot\eta\|_{H^{\frac{3}{2}}_\sharp(0,L)}. \tag{56}$$

with a constant C depending only on the fixed geometry. We set $\bar u = \hat u - u_{\dot\eta}$. This new velocity satisfies:

$$-\operatorname{div}[(A_h\nabla)\bar u] + (B_h\nabla)\hat p_0 = \bar f, \quad \text{in } \Omega_1, \tag{57}$$

$$\operatorname{div}(B_h^\top \bar u) = \bar g, \quad \text{in } \Omega_1, \tag{58}$$

with

$$\bar f := \tilde f + \operatorname{div}[(A_h\nabla)u_{\dot\eta}], \quad \bar g := \tilde g - B_h^\top : \nabla u_{\dot\eta},$$

completed with boundary conditions:

$$\bar u(x,1) = 0, \qquad \forall x \in (0,L), \tag{59}$$

$$\bar u(x,0) = 0, \qquad \forall x \in (0,L). \tag{60}$$

As underlined previously, thanks to the regularity of A_h and B_h, the new source terms $(\bar f, \bar g)$ belong to $L^2_\sharp(\Omega_1) \times H^1_\sharp(\Omega_1)$ and satisfy the following estimates:

$$\|\bar f\|_{L^2_\sharp(\Omega_1)} + \|\bar g\|_{H^1_\sharp(\Omega_1)} \leqq K\left(\|\tilde f\|_{L^2_\sharp(\Omega_1)} + \|\tilde g\|_{H^1_\sharp(\Omega_1)} + \|\dot\eta\|_{H^{\frac{3}{2}}_\sharp(0,L)}\right), \tag{61}$$

where K depends only on R_0. Moreover the average of $\bar g$ on Ω_1 is still equal to zero, since

$$\int_{\Omega_1} B_h^\top : \nabla u_{\dot\eta} = \int_{\Omega_1} \operatorname{div}\,(B_h^\top u_{\dot\eta}) = \int_0^L \dot\eta = 0.$$

Recalling that $u_{\dot\eta}$ satisfies (56), we obtain now that the proof of Lemma 1 reduces to the study of the case $\dot\eta = 0$ (that is solving system (57)–(58)–(59)–(60)).

Third step: $H^1 \times L^2$ estimates. We first define $H^{-1}_\sharp(\Omega_1)$ as the dual space of the subset of $H^1_\sharp(\Omega_1)$ of functions with zero trace on $(0,L)\times\{0\}$ and $(0,L)\times\{1\}$. The aim of this step is to prove that, for any $(\bar f, \bar g) \in H^{-1}_\sharp(\Omega_1) \times L^2_{\sharp,0}(\Omega_1)$, there exists a unique $(\bar u, \hat p_0) \in H^1_\sharp(\Omega_1) \times L^2_{\sharp,0}(\Omega_1)$ solution of (57)–(60) and satisfying

$$\|\bar u\|_{H^1_\sharp(\Omega_1)} + \|\hat p_0\|_{L^2_\sharp(\Omega_1)} \leqq K\left(\|\bar f\|_{H^{-1}_\sharp(\Omega_1)} + \|\bar g\|_{L^2_\sharp(\Omega_1)}\right), \tag{62}$$

where K depends only on R_0. Since the arguments are quite standard (see, for instance, [22,25] in similar contexts), we only sketch the main points of the proof. First, we notice that:

– $A_h \in L^\infty(\Omega_1)$ and there exists two non negative constants α_1 and α_2 controlled by above and from below by a function of R_0 for which

$$\alpha_1 \mathrm{I} \leqq A_h(x,z) \leqq \alpha_2 \mathrm{I}, \quad \forall (x,z) \in \Omega_1,$$

in the sense of symmetric matrices;

– B_h is invertible and B_h^{-1} belongs to $H^1_\sharp((0,L); H^s(0,1))$, for any $s \geqq 0$, with norms dominated by a function of R_0 only.

With the second point at-hand, we build a lifting operator for the divergence. Namely, for any $\chi \in L^2_{\sharp,0}(\Omega_1)$, there exists a vector-field $w \in H^1_\sharp(\Omega_1)$, with $w_{|z=1} = w_{|z=0} = 0$, such that

$$\mathrm{div}\,(B_h^\top w) = \chi, \quad \|w\|_{H^1_\sharp(\Omega_1)} \leqq K\|\chi\|_{L^2_\sharp(\Omega_1)}, \tag{63}$$

where K depends only on R_0. Indeed, as χ has zero average on Ω_1, there exists $v \in H^1_\sharp(\Omega_1)$, with $v_{|z=1} = v_{|z=0} = 0$, such that

$$\mathrm{div}\,(v) = \chi, \quad \|v\|_{H^1_\sharp(\Omega_1)} \leqq C\|\chi\|_{L^2_\sharp(\Omega_1)}.$$

See, for instance, [15, Lemma III.3.1]. We set then $w = B_h^{-\top} v$. As $B_h^{-\top}$ is a multiplier of H^1 with norm bounded by a function of R_0, we obtain (63).

Then, to solve (57)–(60) we first lift the divergence source term $\bar{g}$ by applying the previous construction. We then solve the Stokes-like system (57) and (58) by reproducing the classical arguments for the Stokes system. As A_h satisfies the first point, we first construct a weak solution $\bar{u} \in H^1_\sharp(\Omega_1)$ depending continuously on $(\bar{f}, \bar{g})$. Then, as B_h^{-1} satisfies the second point, we obtain also the pressure $\hat{p}_0 \in L^2_{\sharp,0}(\Omega_1)$ which completes (62).

Fourth step: Proof of Lemma 1, H^2/H^1-regularity. To complete the proof of Lemma 1, it remains to obtain an estimate on the second order derivatives of $\bar{u}$ and the first order derivatives of $\hat{p}_0$. We obtain that

$$\|\bar{u}\|_{H^2_\sharp(\Omega_1)} + \|\hat{p}_0\|_{H^1_\sharp(\Omega_1)} \leqq K\left(\|\bar{f}\|_{L^2_\sharp(\Omega_1)} + \|\bar{g}\|_{H^1_\sharp(\Omega_1)}\right), \tag{64}$$

where K depends only on R_0. We follow the method introduced in [49] and already applied in [22] in our subgraph framework. Thanks to a classical regularization argument, we assume in what follows that $h \in C^\infty_\sharp(0,L)$. In this case, classical elliptic estimates ensure that $(\bar{u}, \hat{p}_0) \in H^2_\sharp(\Omega_1) \times H^1_\sharp(\Omega_1)$. Nervertheless the standard elliptic estimates involve norms of the deformation in $W^{2,\infty}(0,L)$ (see for instance [2]). Consequently, we aim to show that the constant only involves R_0.

First we obtain estimate on $\bar{u}_x := \partial_x \bar{u}$ and $\hat{p}_x := \partial_x \hat{p}_0$. For this purpose, we differentiate the equations (57), (58) satisfied by $(\bar{u}, \hat{p}_0)$ with respect to x. We obtain that $(\bar{u}_x, \hat{p}_x) \in H^1_\sharp(\Omega_1) \times L^2_{\sharp,0}(\Omega_1)$ is the solution of

$$\begin{aligned} -\mathrm{div}[(A_h\nabla)\bar{u}_x] + (B_h\nabla)\hat{p}_x &= \bar{f}_x, \quad \text{on } \Omega_1,\\ \mathrm{div}\,(B_h^\top \bar{u}_x) &= \partial_x \bar{g} - \partial_x B_h^\top : \nabla\bar{u}, \quad \text{on } \Omega_1, \end{aligned}$$

where $\bar{f}_x = \partial_x \bar{f} + \mathrm{div}[(\partial_x A_h \nabla)\bar{u}] - (\partial_x B_h\nabla)\hat{p}_0$, completed with periodic boundary conditions on lateral boundaries of Ω_1 and homogeneous boundary conditions on $y = 1$ and $y = 0$ (we recall that we consider the case $\dot{\eta} = 0$).

We note that

$$\partial_x B_h^\top : \nabla\bar{u} = \mathrm{div}(\partial_x B_h^\top \bar{u}),$$

which implies

$$\int_{\Omega_1} \partial_x B_h^\top : \nabla \bar{u} = 0.$$

Consequently, taking into account that $\bar{g}$ is L-periodic with respect to x, we obtain that $\bar{g}_x = \partial_x \bar{g} - \partial_x B_h^\top : \nabla \bar{u}$ has a zero average on Ω_1. Due to the regularity of $(\bar{u}, \hat{p}_0)$, the right-hand side $(\bar{f}_x, \bar{g}_x)$ belongs to $H_\sharp^{-1}(\Omega_1) \times L_\sharp^2(\Omega_1)$, but we need sharp estimate to show our main result. As we stated previously $(\partial_x A_h, \partial_x B_h) \in L^2((0, L), ; H^s(0, 1))$, for arbitrary $s \geqq 0$, (with norms bounded by a function of R_0) and $H_\sharp^1((0, L) \times (0, 1)) \subset L_\sharp^\infty((0, L); L^2(0, 1))$. Hence, in the spirit of [22, Lemma 6], we obtain

$$\begin{aligned} &\|\mathrm{div}\,((\partial_x A_h \nabla)\bar{u})\|_{H_\sharp^{-1}(\Omega_1)} \\ &\quad \leqq \|(\partial_x A_h \nabla)\bar{u}\|_{L_\sharp^2(\Omega_1)} \leqq K \|\bar{u}\|_{H_\sharp^1(\Omega_1)}^{1/2} \|\bar{u}_x\|_{H_\sharp^1(\Omega_1)}^{1/2}, \end{aligned} \tag{65}$$

and

$$\|\partial_x B_h : \nabla \bar{u}\|_{L_\sharp^2(\Omega_1)} \leqq K \|\bar{u}\|_{H_\sharp^1(\Omega_1)}^{1/2} \|\bar{u}_x\|_{H_\sharp^1(\Omega_1)}^{1/2}, \tag{66}$$

where K depends on R_0. Next we have to estimate $(\partial_x B_h \nabla)\hat{p}_0$ in $H_\sharp^{-1}(\Omega_1)$. Thanks to the Piola identity and the fact that B_h is the cofactor matrix of the gradient of χ_h, we obtain, for any $w \in H_\sharp^1(\Omega_1)$ such that $w_{|z=0} = w_{|z=1} = 0$

$$\int_{\Omega_1} (\partial_x B_h \nabla)\hat{p}_0 w = -\int_{\Omega_1} \hat{p}_0 \partial_x B_h^\top : \nabla w.$$

Consequently, as in the computations of the latter bounds, we obtain:

$$\|(\partial_x B_h \nabla)\hat{p}_0\|_{H_\sharp^{-1}(\Omega_1)} \leqq K \|\hat{p}_0\|_{L_\sharp^2(\Omega_1)}^{1/2} \|\hat{p}_x\|_{L_\sharp^2(\Omega_1)}^{1/2}. \tag{67}$$

We can now apply the result obtained at the previous step to $(\bar{u}_x, \hat{p}_x)$. Combining with (65)–(67), this leads to

$$\begin{aligned} \|\bar{u}_x\|_{H_\sharp^1(\Omega_1)} + \|\bar{p}_x\|_{L_\sharp^2(\Omega_1)} &\leqq K\Bigg(\|\bar{f}\|_{L_\sharp^2(\Omega_1)} + \|\bar{g}\|_{H_\sharp^1(\Omega_1)} \\ &+ \|\bar{u}\|_{H_\sharp^1(\Omega_1)}^{1/2} \|\bar{u}_x\|_{H_\sharp^1(\Omega_1)}^{1/2} + \|\hat{p}_0\|_{L_\sharp^2(\Omega_1)}^{1/2} \|\hat{p}_x\|_{L_\sharp^2(\Omega_1)}^{1/2} | \Bigg), \end{aligned}$$

and finally to:

$$\|\bar{u}_x\|_{H_\sharp^1(\Omega_1)} + \|\bar{p}_x\|_{L_\sharp^2(\Omega_1)} \leqq K \left(\|\bar{f}\|_{L_\sharp^2(\Omega_1)} + \|\bar{g}\|_{H_\sharp^1(\Omega_1)} \right). \tag{68}$$

To obtain a similar estimate on the full second order gradient of $\bar{u}$ (resp. on the full gradient of $\hat{p}_0$), we have to bound $\partial_{zz}\bar{u}$ (resp. $\partial_z \hat{p}_0$). To this end, we note that, differentiating (58) with respect to z and applying (68), we have

$$\| - zh' \partial_{zz}\bar{u}_1 + \partial_{zz}\bar{u}_2\|_{L_\sharp^2(\Omega_1)} \leqq K \left(\|\bar{f}\|_{L_\sharp^2(\Omega_1)} + \|\bar{g}\|_{L_\sharp^2(\Omega_1)} \right).$$

Combining the first equation of (57) with the second equation of (57) multiplied by zh', in order to eliminate the pressure, leads to

$$\|zh'\partial_{zz}\bar{u}_2 + \partial_{zz}\bar{u}_1\|_{L^2_\sharp(\Omega_1)} \leqq K\left(\|\bar{f}\|_{L^2_\sharp(\Omega_1)} + \|\bar{g}\|_{H^1_\sharp(\Omega_1)}\right).$$

By simple algebraic combinations, since $1 + z^2(h')^2 > 0$, we obtain

$$\|\nabla^2\bar{u}\|_{L^2_\sharp(\Omega_1)} \leqq K\left(\|\tilde{f}\|_{L^2_\sharp(\Omega_1)} + \|\bar{g}\|_{L^2_\sharp(\Omega_1)}\right).$$

A similar inequality holds for $\|\partial_z\tilde{p}\|_{L^2_\sharp(\Omega_1)}$, using once again the first equation of (57). Combining these inequalities, we finally obtain the desired bound

$$\|\bar{u}\|_{H^2_\sharp(\Omega_1)} + \|\hat{p}_0\|_{H^1_\sharp(\Omega_1)} \leqq K\left(\|\bar{f}\|_{L^2_\sharp(\Omega_1)} + \|\bar{g}\|_{L^2_\sharp(\Omega_1)}\right).$$

This ends the proof of Lemma 1.

For the study of the whole coupled system, we need an estimate on the surface load applied by the fluid on the structure. If we compute $\phi(u, p, h)$ through the change of variable (27) (that is with respect to $\hat{u}$, $\hat{p}$ and h,) and we use, for instance, the multiplier Lemma [27, Proposition B.1] or Proposition 1, we can also obtain the following corollary, stated without proof:

Corollary 2. *Let $h \in H^2_\sharp(0, L)$ such that $h^{-1} \in L^\infty_\sharp(0, L)$ be given there exists a non decreasing function $K^b : [0, \infty) \to (0, \infty)$ such that, if $\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} \leqq R_0$ the following propositions hold true.*

Given source terms $(f, g) \in L^2_\sharp(\Omega_h) \times (H^1_\sharp(\Omega_1) \cap L^2_{\sharp,0}(\Omega_h))$ and a boundary condition $\dot{\eta} \in H^{\frac{3}{2}}_\sharp(0, L) \cap L^2_{\sharp,0}(0, L)$ satisfying (46), the unique pair $(u, p_0) \in H^2_\sharp(\Omega_h) \times (H^1_\sharp(\Omega_h) \cap L^2_{\sharp,0}(\Omega_h))$ solution to (42)–(43)–(44)–(45) satisfies:

$$\|\phi(u, p_0, h)\|_{H^{\frac{1}{2}}_\sharp(0,L)} \leqq K^b(R_0)\left(\|f\|_{L^2_\sharp(\Omega_h)} + \|g\|_{H^1_\sharp(\Omega_h)} + \|\dot{\eta}\|_{H^{\frac{3}{2}}_\sharp(0,L)}\right). \tag{69}$$

The constant c defined by $c = \frac{1}{L}\int_0^L \phi(u, p_0, h)$ satisfies

$$|c| \leqq K^b(R_0)\left(\|f\|_{L^2_\sharp(\Omega_h)} + \|g\|_{H^1_\sharp(\Omega_h)} + \|\dot{\eta}\|_{H^{\frac{3}{2}}_\sharp(0,L)}\right). \tag{70}$$

4. Proof of Theorem 1

Let $(h^0, \dot{h}^0, u^0) \in H^3_\sharp(0, L) \times H^1_\sharp(0, L) \times H^1_\sharp(\mathcal{F}^0)$ be given and satisfy the compatibility conditions (29)–(32). We consider (h, u, p, c), the associated non-extendable strong solution to (BF) completed with the initial conditions (12)–(14) (in the sense of Definition 1). This solution is defined on some time-interval $[0, T^*)$, where $T^* > 0$. We compute estimates satisfied by this solution on $[0, T]$ for arbitrary $T < T^*$.

The proof of Theorem 1 is divided into three parts, each of them corresponding to the derivation of one estimate similar to (22), (24) and (25) respectively. First, we recall the energy estimate satisfied by the solution, then we prove a distance estimate which ensures that the beam does not touch the bottom of the fluid cavity on the time interval $(0, T)$ and finally we derive a regularity estimate which garantees that the strong solution can be extended on any given time interval, leading to our global-in-time existence theorem.

4.1. Energy Estimate

We first recall the classical estimate associated with the dissipative equations that we consider. We introduce $\mathcal{E}_c$ and $\mathcal{H}$ respectively the total energy of the coupled system and the dissipated energy:

$$\mathcal{E}_c(t) := \frac{1}{2}\left[\int_0^L \left(\rho_s|\partial_t h|^2 + \alpha|\partial_{xx}h|^2 + \beta|\partial_x h|^2\right) + \int_{\mathcal{F}(t)} \rho_f|u|^2\right],$$

$$\mathcal{H}(t) := \gamma\int_0^L |\partial_{tx}h|^2 + \mu\int_{\mathcal{F}(t)} |\nabla u|^2.$$

We then have:

Proposition 2. *The following energy balance holds true*

$$\mathcal{E}_c(t) + \int_0^t \mathcal{H}(s)\mathrm{d}s = \mathcal{E}_c(0), \quad \forall t \in [0, T]. \tag{71}$$

Proof. Multiplying first (1) by u and integrating by parts leads to

$$\begin{aligned}\int_0^t\int_{\mathcal{F}(s)} (\partial_t u + u\cdot\nabla u)\cdot u &= \int_0^t\int_{\mathcal{F}(s)} \mathrm{div}\sigma(u,p)\cdot u\\ &= \int_0^t\int_{\partial\mathcal{F}(s)} \sigma(u,p)n\cdot u - 2\mu\int_0^t\int_{\mathcal{F}(s)} |D(u)|^2\\ &= -\int_0^t\int_0^L \phi(u,p,h)\partial_t h - 2\mu\int_0^t\int_{\mathcal{F}(s)} |D(u)|^2.\end{aligned} \tag{72}$$

In the last equality we have used the coupling conditions at the interface between the fluid and the structure. Moreover thanks to the only vertical motion of the beam together with the divergence free constraint (see [5, Lemma 6] in the three dimensional case), there holds:

$$2\int_{\mathcal{F}(s)} |D(u)|^2 = \int_{\mathcal{F}(s)} |\nabla u|^2. \tag{73}$$

Furthermore, since the boundaries of $\mathcal{F}(t)$ move with the velocity-field u, we have

$$\int_0^t\int_{\mathcal{F}(s)} (\partial_t u + u\cdot\nabla u)\cdot u = \frac{1}{2}\left[\int_{\mathcal{F}(s)} |u|^2\right]_{s=0}^{s=t}, \tag{74}$$

Consequently

$$\frac{1}{2}\int_{\mathcal{F}(t)} |u|^2 + \mu \int_0^t \int_{\mathcal{F}(s)} |\nabla u|^2 = \frac{1}{2}\int_{\mathcal{F}^0} |u^0|^2 - \int_0^t \int_0^L \phi(u,p,h)\partial_t h. \quad (75)$$

If we now multiply the beam equation (3) by $\partial_t h$ we obtain, after time and space integration by parts,

$$\frac{1}{2}\left[\int_0^L \left(\rho_s|\partial_t h|^2 + \alpha|\partial_{xx}h|^2 + \beta|\partial_x h|^2\right)\right]_{s=0}^{s=t} + \gamma\int_0^t\int_0^L |\partial_{tx}\eta|^2 = \int_0^t\int_0^L \phi(u,p,h)\partial_t h. \quad (76)$$

By summing (75) and (76), we obtain the expected result. □

4.2. Distance Estimate

The aim of this section is to prove the following proposition:

Proposition 3. *There exists a constant C_0 depending only on initial data for which:*

$$\sup_{t\in(0,T)}\left(\gamma\|h(t,\cdot)\|^2_{H^2_\sharp(0,L)} + \|h^{-1}(t,\cdot)\|_{L^1_\sharp(0,L)}\right) + \alpha\int_0^T \|h\|^2_{H^3_\sharp(0,L)} \leqq C_0(1+T).$$

The remainder of this paragraph is devoted to the proof of this result. Let us consider $0 \leqq t \leqq T$. In all what follows C_0 denotes a constant depending only on the initial data but which may change between lines. Let $w = \nabla^\perp\psi = (-\partial_y\psi, \partial_x\psi)$ where

$$\psi(x,y,t) = \partial_x h(x,t)\chi_0\left(\frac{y}{h(x,t)}\right), \quad \forall (x,y,t)\in\mathcal{Q}_T,$$

with

$$\chi_0(z) = z^2(3-2z), \quad \forall z\in(0,1).$$

Combining the regularity of h (which implies that $h \in C([0,T]; H^3_\sharp(0,L)) \cap H^1(0,T; H^2_\sharp(0,L))$ thanks to [40, Theorem 3.1] see (38)) with $\chi_0 \in C^\infty([0,1])$ we obtain that $w \in H^1(\mathcal{Q}_T)$ and $\nabla^2 w \in L^2(\mathcal{Q}_T)$. This regularity is enough to justify all computations below as (h,u,p,c) is a strong solution. Moreover, w is divergence free by construction and, since $\chi_0(1) = 0$ and $\chi_0'(1) = \chi_0(0) = \chi_0'(0) = 0$, there holds:

$$\begin{aligned} w(x,h(t,x),t) &= \partial_{xx}h(t,x)e_2, && \text{for all } x\in(0,L) \text{ and } t\in(0,T),\\ w(x,0,t) &= 0, && \text{for all } x\in(0,L) \text{ and } t\in(0,T). \end{aligned}$$

Consequently, we multiply (1) by w and (3) by $\partial_{xx}h$ and integrate on $\mathcal{Q}_t$ for arbitrary $t < T$. We get after integration by parts (note that the terms involving the fluid/beam interactions cancel out since the structure test function is the trace of the fluid test function on the interface):

$$-\int_{\mathcal{Q}_t} \rho_f(\partial_t u + u\cdot\nabla u)\cdot w - 2\mu\int_{\mathcal{Q}_t} D(u) : D(w)$$
$$+\int_0^t\int_0^L \left(\beta|\partial_{xx}h|^2 + \alpha|\partial_{xxx}h|^2\right)$$
$$+\left[\int_0^L \left(\frac{\gamma}{2}|\partial_{xx}h|^2 - \rho_s\partial_t h\,\partial_{xx}h\right)\right]_{s=0}^{s=t} = \int_0^t\int_0^L \rho_s|\partial_{tx}h|^2. \quad (77)$$

We first show that this identity leads to an estimate that is comparable to (23) up to remainder terms we shall bound afterwards. The term that will enable us to bound h^{-1} is $2\mu\int_{\mathcal{Q}_t} D(u) : D(w)$. To deal with this term, we introduce a well chosen pressure:

$$\begin{cases} q(x,y,t) := q_s(x,t) + \partial_{xy}\psi(x,y,t), \\ q_s(x,t) := -\displaystyle\int_0^x \partial_{yyy}\psi(s,y,t)\,\mathrm{d}s, \quad \forall (x,y,t)\in\mathcal{Q}_T. \end{cases} \quad (78)$$

An easy computation gives $\partial_{yyy}\psi(s,y,t) = -12\dfrac{\partial_x h(x,t)}{(h(x,t))^3}$, so that q_s satisfies

$$q_s(x,t) = 12\int_0^x \frac{\partial_x h(s,t)}{(h(s,t))^3}\,\mathrm{d}s = 6\left[\frac{1}{|h(0,t)|^2} - \frac{1}{|h(x,t)|^2}\right]. \quad (79)$$

In particular, q_s does not depend on y. Furthermore $\nabla q \in L^2(\mathcal{Q}_T)$. Applying again the fact that w is divergence-free, we obtain:

$$\begin{aligned} 2\int_{\mathcal{Q}_t} D(u) : D(w) &= \int_{\mathcal{Q}_t} (2D(w) - qI_2) : D(u) \\ &= \int_0^t\int_{\partial\mathcal{F}(s)} (2D(w) - qI_2)n\cdot u \\ &\quad - \int_{\mathcal{Q}_t} (\Delta w - \nabla q)\cdot u. \end{aligned} \quad (80)$$

Note, that all the terms make sense thanks to the regularity of (w,q). In this last identity, by definition (78) of q, we have:

$$\begin{aligned} \int_{\mathcal{Q}_t} (\Delta w - \nabla q)\cdot u &= \int_{\mathcal{Q}_t} \partial_{xxx}\psi\, u_2 - 2\partial_{yxx}\psi\, u_1 \\ &= \int_0^t\int_{\partial\mathcal{F}(s)} (n_1\partial_{xx}\psi u_2 - 2n_2\partial_{xx}\psi u_1) \\ &\quad - \int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1) \\ &= -\int_0^t\int_0^L \partial_{xx}\psi(x,h(x,s),s)\partial_t h(x,s)\partial_x h(x,s)\,\mathrm{d}x\mathrm{d}s \\ &\quad - \int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1). \end{aligned}$$

Similarly, the other term of (80) can be expressed as

$$\begin{aligned}\int_0^t \int_{\partial\mathcal{F}(s)} (2D(w) - qI_2)n \cdot u \\ = \int_0^t \int_0^L \partial_t h(x,t)\Big((\partial_{yy}\psi(x,h(x,s),s) - \partial_{xx}\psi(x,h(x,s),s))\partial_x h(x,s) \\ + \partial_{yx}\psi(x,h(x,s),s) - q_s(x,s)\Big)\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Differentiating the identity $\partial_y\psi(x,h(x,s),s) = 0$ (that holds true since $\chi_0'(1) = 0$), with respect to x, yields

$$\partial_{yx}\psi(x,h(x,s),s) + \partial_x h(x,s)\partial_{yy}\psi(x,h(x,s),s) = 0.$$

Consequently, we simplify:

$$\begin{aligned}\int_0^t \int_{\partial\mathcal{F}(s)} (2D(w) - qI_2)n \cdot u \\ = -\int_0^t \int_0^L \partial_t h(x,t)\partial_{xx}\psi(x,h(x,s),s))\partial_x h(x,s)\,\mathrm{d}x\mathrm{d}s \\ - \int_0^t \int_0^L \partial_t h(x,s) q_s(x,s)\,\mathrm{d}x\mathrm{d}s.\end{aligned}$$

Combining the computations of both terms in (80), we obtain finally:

$$2\int_{\mathcal{Q}_t} D(u) : D(w) = -\int_0^t \int_0^L \partial_t h q_s + \int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1). \tag{81}$$

At this point, we replace q_s by its explicit value (see (79)) and by remembering that the average of $\partial_t h$ is zero, we obtain

$$\int_0^t \int_0^L \partial_t h q_s = -6\int_0^t \int_0^L \frac{\partial_t h}{|h|^2} = \left[\int_0^L \frac{6}{h(x,s)}\,\mathrm{d}x\right]_{s=0}^{s=t}. \tag{82}$$

Consequently, from (81), (82), the equality (77) reduces to:

$$\begin{aligned}\left[\int_0^L \left(\frac{\gamma}{2}|\partial_{xx}h|^2 - \rho_s \partial_t h\,\partial_{xx}h + \frac{6\mu}{h}\right)\right]_{s=0}^{s=t} + \int_0^t \int_0^L \left(\beta|\partial_{xx}h|^2 + \alpha|\partial_{xxx}h|^2\right) \\ = \int_0^t \int_0^L \rho_s|\partial_{tx}h|^2 + \mu\int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1) + \int_{\mathcal{Q}_t} \rho_f(\partial_t u + u\cdot\nabla u)\cdot w.\end{aligned} \tag{83}$$

We recognize in the left-hand side of this equality the quantities that we want to estimate as in (23). Compared to (23), we have two additional terms

$$T_1 = \mu\int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1), \quad T_2 = \int_{\mathcal{Q}_t} \rho_f(\partial_t u + u\cdot\nabla u)\cdot w.$$

To bound these terms, we need precise estimates on the stream-function ψ that are gathered in "Appendix A.2".

First, T_1 is bounded by applying Proposition 8 and energy estimate (71)

$$\begin{aligned}\left|\int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1)\right| &\leqq C\|\partial_{xx}\psi\|_{L^2(\mathcal{Q}_t)}\|\nabla u\|_{L^2(\mathcal{Q}_T)},\\ &\leqq C_0\Big(\int_0^t \Big[\|h\|_{L^\infty_\sharp(0,L)}\|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}\\ &\quad +\|\partial_{xx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)}\|\partial_{xxx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)}\Big]\Big)^{\frac{1}{2}},\end{aligned}$$

with a constant C_0 depending only on the initial data. Once again, using the energy estimate (71), we obtain

$$\begin{aligned}&\left|\int_{\mathcal{Q}_t} \partial_{xx}\psi(\partial_x u_2 - 2\partial_y u_1)\right|\\ &\leqq C_0\Bigg[\left(\sup_{t\in(0,T)}\|h\|^{\frac{1}{2}}_{L^\infty_\sharp(0,L)}\right)\left(\int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}\right)^{\frac{1}{2}}\\ &\quad +\left(\sup_{t\in(0,T)}\|\partial_{xx}h\|^{\frac{3}{4}}_{L^2_\sharp(0,L)}\right)\left(\int_0^t \|\partial_{xxx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)}\right)^{\frac{1}{2}}\Bigg]\\ &\leqq C_0(1+T) + \varepsilon_1\int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)},\end{aligned}\tag{84}$$

for arbitrary small $\varepsilon_1 > 0$, that will be chosen later on. We note that C_0 depends on ε_1 a priori but the value of this parameter will be fixed to a universal constant afterwards. This remark is also valid when other ε's are introduced.

Concerning T_2, taking into account the convection of the fluid domain by the fluid velocity, we have

$$\begin{aligned}T_2 &= \int_{\mathcal{Q}_t}(\partial_t u + u\cdot\nabla u)\cdot w\\ &= \left[\int_{\mathcal{F}(s)} u(\cdot,s)\cdot w(\cdot,s)\right]_{s=0}^{s=t} - \int_{\mathcal{Q}_t}\partial_t w - \int_{\mathcal{Q}_t} u\cdot\nabla w\cdot u.\end{aligned}\tag{85}$$

We bound the first term on the right-hand side of (85) using a Cauchy–Schwarz inequality. Applying the energy estimate (71) and the estimates (125)–(126) on the gradient of the stream-function, one gets

$$\begin{aligned}\left|\left[\int_{\mathcal{F}(s)} u(\cdot,s)\cdot w(\cdot,s)\right]_{s=0}^{s=t}\right| &\leqq C_0 + \|u(\cdot,t)\|_{L^2(\mathcal{F}(t))}\|\nabla\psi(\cdot,t)\|_{L^2(\mathcal{F}(t))}\\ &\leqq C_0\left(1+\left[\int_0^L \frac{1}{h}\right]^{\frac{1}{4}}\right)\\ &\leqq C_0 + \varepsilon_2\int_0^L \frac{1}{h},\end{aligned}\tag{86}$$

for arbitrary small $\varepsilon_2 > 0$.

For the second term of the right-hand side of (85), we first integrate by parts in space. Since $\psi(x,h(x,s),s) = \partial_x h(x,s)$ and $\partial_y \psi(x,h(x,s),s) = 0$, we have $\partial_t \psi(x,h(x,s),s) = \partial_{tx} h(x,s)$ so that:

$$\begin{aligned}\int_{\mathcal{Q}_t} u \cdot \partial_t w &= -\int_0^t \int_0^L \partial_t h(x,s)\partial_x h(x,s)\partial_t \psi(x,h(x,s),s)\,\mathrm{d}x\mathrm{d}s \\ &\quad + \int_{\mathcal{Q}_t} (\partial_y u_1 - \partial_x u_2)\partial_t \psi \\ &= -\int_0^t \int_0^L \partial_t h \partial_x h \partial_{tx} h + \int_{\mathcal{Q}_t} (\partial_y u_1 - \partial_x u_2)\partial_t \psi.\end{aligned}$$

Applying the energy estimate (71) and using the following one dimensional embedding inequality

$$\|\partial_x h(\cdot,t)\|_{L^\infty_\sharp(0,L)} \leqq C\|\partial_{xx} h(\cdot,t)\|_{L^2_\sharp(0,L)},$$

we can estimate the boundary term

$$\begin{aligned}\left|\int_0^t \int_0^L \partial_t h \partial_x h \partial_{tx} h\right| &\leqq \int_0^t \|\partial_t h\|_{L^2_\sharp(0,L)} \|\partial_x h\|_{L^\infty_\sharp(0,L)} \|\partial_{tx} h\|_{L^2_\sharp(0,L)} \\ &\leqq C\int_0^t \|\partial_{xx} h\|_{L^2_\sharp(0,L)} \|\partial_{tx} h\|^2_{L^2_\sharp(0,L)} \leqq C_0. \end{aligned} \tag{87}$$

Taking into account (127) and the energy estimate (71), we now bound the second term by

$$\begin{aligned}\left|\int_{\mathcal{Q}_t} (\partial_y u_1 - \partial_x u_2)\partial_t \psi\right| &\leqq C_0 \|\partial_t \psi\|_{L^2(\mathcal{Q}_t)} \\ &\leqq C_0 \left[\int_0^t \left(\|\partial_{tx} h\|^2_{L^2_\sharp(0,L)} + \|\partial_{xxx} h\|_{L^2_\sharp(0,L)}\right)\right]^{\frac{1}{2}} \\ &\leqq C_0(1+T) + \varepsilon_3 \int_0^t \|\partial_{xxx} h\|^2_{L^2_\sharp(0,L)}, \end{aligned} \tag{88}$$

where $\varepsilon_3 > 0$ will be chosen later on.

Finally for the last term in the right hand side of (85), we have, after space integration by parts:

$$\int_{\mathcal{Q}_t} u \cdot \nabla w \cdot u = \int_0^t \int_0^L |\partial_t h|^2 \partial_{xx} h - \int_{\mathcal{Q}_t} u \cdot \nabla u \cdot w. \tag{89}$$

Concerning the boundary integral in (89), we apply (71) to show the following estimate

$$\begin{aligned}\left|\int_0^t \int_0^L |\partial_t h|^2 \partial_{xx} h\right| &\leqq C\int_0^t \left(\|\partial_{xx} h\|_{L^2_\sharp(0,L)} \|\partial_t h\|^2_{L^\infty_\sharp(0,L)}\right) \\ &\leqq \sup_{t\in(0,T)} \|\partial_{xx} h\|_{L^2_\sharp(0,L)} \quad \int_0^t \|\partial_{tx} h\|^2_{L^2_\sharp(0,L)} \leqq C_0. \end{aligned} \tag{90}$$

Moreover, for the volume integral in the right hand side of (89), we have

$$\left|\int_{\mathcal{Q}_t} u\cdot\nabla u\cdot w\right| \leqq \int_0^t\int_0^L\left(\left(\int_0^{h(x,s)}|u|^2\right)^{\frac12}\left(\int_0^{h(x,s)}|\nabla u|^2\right)^{\frac12}\sup_{y\in(0,h(x,s))}|w(x,y,s)|\right)\mathrm{d}x\mathrm{d}s.$$

From (124) we know that the following pointwise estimate holds true:

$$|w(x,y,s)|\leqq C\left(|\partial_{xx}h(x,s)|+\frac{|\partial_x h(x,s)|}{h(x,s)}+\frac{|\partial_x h(x,s)|^2}{h(x,s)}\right),\quad \forall\,(x,y)\in\mathcal{F}(s).$$

Thus we define

$$I_1=\int_0^t\int_0^L\left(\left(\int_0^{h(x,s)}|u|^2\right)^{\frac12}\left(\int_0^{h(x,s)}|\nabla u|^2\right)^{\frac12}|\partial_{xx}h(x,s)|\right)\mathrm{d}x\mathrm{d}s,$$

$$I_2=\int_0^t\int_0^L\left(\left(\int_0^{h(x,s)}|u|^2\right)^{\frac12}\left(\int_0^{h(x,s)}|\nabla u|^2\right)^{\frac12}\frac{|\partial_x h(x,s)|}{h(x,s)}\right)\mathrm{d}x\mathrm{d}s$$

and

$$I_3=\int_0^t\int_0^L\left(\left(\int_0^{h(x,s)}|u|^2\right)^{\frac12}\left(\int_0^{h(x,s)}|\nabla u|^2\right)^{\frac12}\frac{|\partial_x h(x,s)|^2}{h(x,s)}\right)\mathrm{d}x\mathrm{d}s.$$

We now take care of each quantity. Applying the one dimensional embedding inequality

$$\|\partial_{xx}h(\cdot,t)\|_{L^\infty_\sharp(0,L)}\leqq C\|\partial_{xxx}h(\cdot,t)\|_{L^2_\sharp(0,L)}$$

and the energy estimate (71), we obtain:

$$\begin{aligned}
I_1&\leqq C\int_0^t\left(\left(\int_{\mathcal{F}(s)}|u|^2\right)^{\frac12}\left(\int_{\mathcal{F}(s)}|\nabla u|^2\right)^{\frac12}\|\partial_{xx}h\|_{L^\infty_\sharp(0,L)}\right)\\
&\leqq C_0\int_0^t\left(\left(\int_{\mathcal{F}(s)}|\nabla u|^2\right)^{\frac12}\|\partial_{xxx}h\|_{L^2_\sharp(0,L)}\right)\\
&\leqq C_0\|\nabla u\|_{L^2_\sharp(\mathcal{Q}_T)}\left[\int_0^t\|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}\right]^{\frac12}\\
&\leqq \frac{C_0}{\varepsilon_4}+\varepsilon_4\int_0^t\|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)},
\end{aligned}$$

for arbitrary small $\varepsilon_4>0$. We now take care of I_2. We note then that u vanishes on $y=0$ so that a Poincaré inequality yields:

$$\left(\int_0^{h(x,s)} |u|^2\right)^{\frac{1}{2}} \leqq Ch(x,s)\left(\int_0^{h(x,s)} |\nabla u|^2\right)^{\frac{1}{2}}.$$

Using this bound to estimate I_2 we get

$$I_2 \leqq C\int_0^t \left(\|\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))}\|\partial_x h\|_{L^\infty_\sharp(0,L)}\right) \leqq C_0$$

as $\|\partial_x h(\cdot,t)\|_{L^\infty_\sharp(0,L)} \leqq C\|\partial_{xx}h(\cdot,t)\|_{L^2_\sharp(0,L)}$, which remains uniformly bounded in time (see (71)). With similar arguments, we also prove $I_3 \leqq C_0$. This yields finally

$$\left|\int_{\mathcal{Q}_t} u\cdot\nabla u\cdot w\right| \leqq C_0 + \varepsilon_4\int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}$$

and thus, taking into account (90)

$$\left|\int_{\mathcal{Q}_t} u\cdot\nabla w\cdot u\right| \leqq C_0 + \varepsilon_4\int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}. \tag{91}$$

Finally, T_2 can be bounded, thanks to (86), (87), (88), (91), as

$$|T_2| \leqq C_0(1+T) + \varepsilon_5\int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)} + \varepsilon_2\int_0^L \frac{1}{h}, \tag{92}$$

where ε_5 and ε_2 are to be chosen small enough.

Combining (84) and (92) to bound the right-hand side of (83) and taking into account (71) to bound the remaining terms on the right-hand side depending on h that might be concerned, we get, for any $t \leqq T$,

$$\begin{aligned}\int_0^L \left(\frac{\gamma}{2}|\partial_{xx}h|^2 + \frac{6\mu}{h}\right) &+ \int_0^t\int_0^L \left(\beta|\partial_{xx}h|^2 + \alpha|\partial_{xxx}h|^2\right)\\ &\leqq C_0(1+T) + \rho_s\|\partial_t h(\cdot,t)\|_{L^2_\sharp(0,L)}\|\partial_{xx}h(\cdot,t)\|_{L^2_\sharp(0,L)}\\ &\quad + \rho_s\int_0^t\int_0^L |\partial_{tx}h|^2 + \varepsilon\left(\int_0^L \frac{6\mu}{h} + \int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}\right)\\ &\leqq C_0(1+T) + \varepsilon\left(\int_0^L \frac{6\mu}{h} + \int_0^t \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)}\right),\end{aligned}$$

for some arbitrary small $\varepsilon > 0$. We conclude the proof of the distance estimate of Proposition 3 by choosing ε small enough.

Remark 4.

- The derived distance estimate relies strongly on the fact that the beam motion is only transverse and that we control the curvature of the elastic boundary. Indeed, we need to have bounds on the deformation h in $L^\infty(0,T;H^2_\sharp(0,L))$ and in $L^2(0,T;H^3_\sharp(0,L))$ to control remainder terms. Both norms are bounded because $\alpha > 0$, even if the first one is controlled via the energy bound while the second one is controlled simultaneously with h^{-1}.

– To prove our distance estimate we need to control $\partial_t h$ in $L^2(0, T; H^1_\sharp(0, L))$. This is the reason why we assume $\gamma > 0$. One may wonder whether the fluid dissipation would be sufficient. A priori, from the $L^2(0, T; H^1_\sharp(\mathcal{F}(t))$ bound of the fluid velocity we only get a control on $\partial_t h$ in the space $L^2(0, T; H^{1/2}_\sharp(0, L))$, which is not enough.

4.3. Regularity Estimate

Combining Proposition 3 with Proposition 7, we obtain that

$$h^{-1} \in L^\infty(0, T; L^1_\sharp(0, L)).$$

More precisely, we have that there exists a non-decreasing function $R : [0, T^*) \to [0, \infty)$ bounded on all bounded subintervals of $[0, T^*)$ such that:

$$\|h^{-1}(\cdot, t)\|_{L^\infty_\sharp(0,L)} + \|h(\cdot, t)\|_{H^2_\sharp(0,L)} \leqq R_t \quad \forall t \in [0, T^*). \tag{93}$$

In particular, the beam never touches the bottom of the fluid cavity on bounded time intervals. This lower bound on h ensures that the elliptic regularity result derived in Section 3.2 applies and enables us to pursue further in order to prove global existence of strong solutions. By choosing appropriate test functions, we obtain the following quantitative estimate:

Proposition 4. *There exists a function* $C^0_{\mathrm{reg}} : [0, T^*) \to [0, \infty)$ *bounded on all bounded subintervals of* $[0, T^*)$ *such that:*

$$\|u(\cdot, t)\|^2_{H^1_\sharp(\mathcal{F}(t))} + \|\partial_t h(\cdot, t)\|^2_{H^1_\sharp(0,L)} + \|h(\cdot, t)\|^2_{H^3_\sharp(0,L)} \leqq C^0_{\mathrm{reg}}(t). \tag{94}$$

The remainder of this subsection is devoted to the proof of this proposition. We fix $T < T^*$ and construct the C^0_{reg} for $t \in [0, T]$. We split the proof into three steps. In the first one, we aim at multiplying the fluid equation and the structure equation by $\partial_t u$ and $\partial_{tt} h$ respectively. However, $\partial_t u$ is not an appropriate multipier of the fluid equation since it does not take into account the motion of the fluid domain. A natural choice is then the total derivative $\partial_t u + u \cdot \nabla u$, but this function is not divergence free, so we introduce a modified divergence free test function following ideas of [9]. This step requires us to bound the fluid velocity in $H^2_\sharp(\mathcal{F}(t))$ and consequently, thanks to the elliptic estimates derived in Section 3.2, the structure velocity in $H^{3/2}_\sharp(0, L)$. As we do not get estimates on this quantity in our first step, we need a second step to obtain a regularity estimate on the deformation of the beam. This second estimate depends itself on the regularity of the applied fluid-force and thus on high-order norms of the fluid-velocity and the pressure. The final step consists in a well chosen combination of the two previous estimates in order to obtain the expected result.

Remark 5.

– As the function $R : [0, \infty) \mapsto (0, \infty)$ satisfies (93), when we apply below Proposition 1, Lemma 1 or Corollary 2, the associated respective constants K^e, K^s or K^b define non-decreasing functions of time bounded on bounded subintervals of $[0, T^*)$.

– In the computations below, we denote by C_0 a constant depending on initial data and by $C_0 : [0, T^*) \to (0, \infty)$ a function that is bounded on all bounded subintervals of $[0, T^*)$. The value of this constant/function may vary between lines.

Step 1 : $L^\infty(0, T; H^1)$ regularity estimate of the fluid and structure velocities
As explained in the previous paragraph, we introduce a vector-field $\hat{\Lambda}(\cdot, t)$ which coincides with u on $\partial\mathcal{F}(t)$ for all $t \in (0, T)$, and which satisfies the following:

– $\hat{\Lambda}$ is divergence free,
– $\nabla\hat{\Lambda} \in L^2_\sharp(\mathcal{Q}_T)$ and $\nabla^2\hat{\Lambda} \in L^2_\sharp(\mathcal{Q}_T)$ with

$$\|\hat{\Lambda}(\cdot, t)\|_{L^2_\sharp(\mathcal{F}(t))} \leqq C_0(t)\|\partial_t h(\cdot, t)\|_{L^2_\sharp(0,L)}, \tag{95}$$

$$\|\nabla\hat{\Lambda}(\cdot, t)\|_{L^2_\sharp(\mathcal{F}(t))} \leqq C_0(t)\|\partial_t h(\cdot, t)\|_{H^1_\sharp(0,L)}, \tag{96}$$

$$\|\nabla^2\hat{\Lambda}(\cdot, t)\|_{L^2_\sharp(\mathcal{F}(t))} \leqq C_0(t)\|\partial_t h(\cdot, t)\|_{H^2_\sharp(0,L)}, \tag{97}$$

for almost every $t \in (0, T)$,
– $\hat{\Lambda} = u$ and $\partial_2\hat{\Lambda} = 0$ on $y = h(x, t)$,
– $\hat{\Lambda} = 0$ on $y = 0$.

The construction of $\hat{\Lambda}$ is given in "Appendix B". With the notations of this appendix, we have $C_0(t) = K^l(R_t)$ that is indeed a function which is bounded on all bounded subintervals of $[0, T^*)$.

Next we define v as

$$v := \partial_t u + \hat{\Lambda} \cdot \nabla u - u \cdot \nabla\hat{\Lambda}.$$

Given $t \leqq T$, it is a suitable multiplier for (1) on $\mathcal{Q}_t$ as it belongs to $L^2_\sharp(\mathcal{Q}_t)$. Indeed, by classical Sobolev embedding, we have

$$\begin{aligned}\|\hat{\Lambda} \cdot \nabla u\|_{L^2(\mathcal{Q}_t)} &\leqq \int_0^t \|\hat{\Lambda}\|_{L^4(\mathcal{F}(s))}\|\nabla u\|_{L^4(\mathcal{F}(s))} \\ &\leqq C_0(t)\int_0^t \|\hat{\Lambda}\|_{H^1(\mathcal{F}(s))}\|u\|_{H^2(\mathcal{F}(s))} \\ &\leqq C_0(t) \sup_{s\in(0,t)} \|\partial_t h(\cdot, s)\|_{H^1_\sharp(0,L)} \\ &\quad \times \left(\|\nabla u\|_{L^2(\mathcal{Q}_t)} + \|\nabla^2 u\|_{L^2(\mathcal{Q}_t)}\right).\end{aligned}$$

We note here that we have used the continuous embedding $H^1(\mathcal{F}(s)) \hookrightarrow L^4(\mathcal{F}(s))$. A priori, the constant associated with this embedding depends on the domain $\mathcal{F}(s)$ and thus on the deformation of the beam. However, going back in a fixed domain and interpolating the results of Proposition 1 (or extrapolating an equivalent version for the L^4-space that we skip for conciseness), we might prove that this constant is uniformly bounded locally in time, as $\|h(\cdot, s)\|_{H^2_\sharp(0,L)}$ and $\|h^{-1}(\cdot, s)\|_{L^\infty_\sharp(0,L)}$ are bounded locally. In the same way, in what follows, we may use interpolation inequalities in $\mathcal{F}(s)$ for which the constant will be bounded by a locally bounded function of $\|h(\cdot,s)\|_{H^2_\sharp(0,L)}$ and $\|h^{-1}(\cdot,s)\|_{L^\infty_\sharp(0,L)}$ and thus by a locally bounded function of R_s. Similarly, we obtain $u\cdot\nabla\hat{\Lambda}\in L^2(\mathcal{Q}_t)$.

Thus, we multiply (1) by v on $\mathcal{Q}_t$ and obtain the following identity

$$\int_{\mathcal{Q}_t}\rho_f(\partial_t u+u\cdot\nabla u)\cdot(\partial_t u+\hat{\Lambda}\cdot\nabla u-u\cdot\nabla\hat{\Lambda})$$
$$=\int_{\mathcal{Q}_t}\mathrm{div}\sigma\cdot(\partial_t u+\hat{\Lambda}\cdot\nabla u-u\cdot\nabla\hat{\Lambda}). \tag{98}$$

Formally, the trace of v on $y=h$ is equal to $\partial_{tt}h$, so we expect that this identity has to be combined by the structure equation multiplied by $\partial_{tt}h$. Again, $\partial_{tt}h$ is a suitable multiplier for (3) as it belongs to $L^2_\sharp((0,L)\times(0,t))$ and we obtain the following identity:

$$\int_0^L\rho_s|\partial_{tt}h|^2-\int_0^L\phi(u,p,h)\partial_{tt}h$$
$$=-\alpha\int_0^L\partial_{xxxx}h\partial_{tt}h+\beta\int_0^L\partial_{xx}h\partial_{tt}h+\gamma\int_0^L\partial_{xxt}h\partial_{tt}h \tag{99}$$

To compute the right-hand sides of (98) and (99), we need more regularity than the one satisfied by the strong solution under consideration. However the following lemma holds true:

Lemma 2. *For any triplet (w,q,b) satisfying the regularity assumptions of Definition 1, namely*

$$b\in H^2(0,T;L^2_\sharp(0,L))\cap L^2(0,T;H^4_\sharp(0,L)),$$
$$w\in H^1_\sharp(\mathcal{Q}_T),\quad \nabla^2 w\in L^2_\sharp(\mathcal{Q}_T),\quad q\in L^2_\sharp(\mathcal{Q}_T),\quad \nabla q\in L^2_\sharp(\mathcal{Q}_T),$$

and such that $w(x,h(x,t),t)=\partial_t b(x,t)e_2\in L^2_{\sharp,0}(0,L)$ and $\mathrm{div}\,w=0$, the following identities are satisfied:

$$\int_{\mathcal{Q}_t}\mathrm{div}\sigma(w,q)\cdot(\partial_t w+\hat{\Lambda}\cdot\nabla w-w\cdot\nabla\hat{\Lambda})=-\int_0^t\int_0^L\phi(w,q,h)\partial_{tt}b$$
$$-\frac{\mu}{2}\int_{\mathcal{F}(t)}|\nabla w|^2(t)+\frac{\mu}{2}\int_{\mathcal{F}(0)}|\nabla w|^2(0)$$
$$-2\mu\int_{\mathcal{Q}_t}D(w):\left([\nabla\hat{\Lambda}]^\top\nabla w+\nabla\hat{\Lambda}[\nabla w]^\top-D(w\cdot\nabla\hat{\Lambda})\right), \tag{100}$$

and

$$\alpha\int_0^t\int_0^L\partial_{xxxx}b\,\partial_{tt}b-\beta\int_0^t\int_0^L\partial_{xx}b\,\partial_{tt}b-\gamma\int_0^t\int_0^L\partial_{xxt}b\,\partial_{tt}b$$
$$=\int_0^L\left(\frac{\gamma}{2}|\partial_{tx}b(x,t)|^2-\beta\partial_t b(x,t)\partial_{xx}b(x,t)-\alpha\partial_{tx}b(x,t)\partial_{xxx}b(x,t)\right)\mathrm{d}x$$
$$-\beta\int_0^t\int_0^L|\partial_{tx}b|^2-\alpha\int_0^t\int_0^L|\partial_{txx}b|^2$$

$$
-\int_0^L \Big(\frac{\gamma}{2}|\partial_{tx} b(x,0)|^2 - \beta \partial_t b(x,0)\partial_{xx} b(x,0) - \alpha \partial_{tx} b(x,0)\partial_{xxx} b(x,0)\Big)\,\mathrm{d}x. \tag{101}
$$

The proof of this lemma relies on regularization arguments and is postponed to "Appendix B".

Thus, we apply (100) for $w = u$ and $b = h$. Then identity (98) becomes

$$
\begin{aligned}
&\int_{\mathcal{Q}_t} \rho_f(\partial_t u + u\cdot\nabla u)\cdot(\partial_t u + \hat{\Lambda}\cdot\nabla u - u\cdot\nabla\hat{\Lambda}) = \\
&\quad -\int_0^t\int_0^L \phi(u,p,h)\partial_{tt}h - \frac{\mu}{2}\int_{\mathcal{F}(t)}|\nabla u|^2 + \frac{\mu}{2}\int_{\mathcal{F}(0)}|\nabla u_0|^2 \\
&\quad -2\mu\int_{\mathcal{Q}_t} D(u) : \Big([\nabla\hat{\Lambda}]^\top\nabla u + \nabla\hat{\Lambda}[\nabla u]^\top - D(u\cdot\nabla\hat{\Lambda})\Big).
\end{aligned} \tag{102}
$$

For the left-hand side of (102), denoted LHS, we have:

$$
\begin{aligned}
\text{LHS} &= \int_{\mathcal{Q}_t}\rho_f|\partial_t u + u\cdot\nabla u|^2 - \int_{\mathcal{Q}_t}\rho_f(\partial_t u + u\cdot\nabla u)\cdot(u\cdot\nabla u) \\
&\quad + \int_{\mathcal{Q}_t}\rho_f\,(\partial_t u + u\cdot\nabla u)\cdot(\hat{\Lambda}\cdot\nabla u - u\cdot\nabla\hat{\Lambda}), \\
&\geqq \frac{1}{2}\int_{\mathcal{Q}_t}\rho_f|\partial_t u + u\cdot\nabla u|^2 - \frac{1}{2}\int_{\mathcal{Q}_t}\rho_f|u\cdot\nabla u|^2 \\
&\quad + \int_{\mathcal{Q}_t}\rho_f\,(\partial_t u + u\cdot\nabla u)\cdot(\hat{\Lambda}\cdot\nabla u - u\cdot\nabla\hat{\Lambda}),
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\frac{\mu}{2}\int_{\mathcal{F}(t)}|\nabla u|^2 + \frac{1}{2}\int_{\mathcal{Q}_t}\rho_f|\partial_t u + u\cdot\nabla u|^2 + \int_0^t\int_0^L\phi(u,p,h)\partial_{tt}h \\
&\quad\leqq \frac{1}{2}\int_{\mathcal{Q}_t}\rho_f|u\cdot\nabla u|^2 - \int_{\mathcal{Q}_t}\rho_f(\partial_t u + u\cdot\nabla u)(\hat{\Lambda}\nabla u - u\cdot\nabla\hat{\Lambda}) \\
&\qquad -2\mu\int_{\mathcal{Q}_t}D(u) : \Big([\nabla\hat{\Lambda}]^\top\nabla u + \nabla\hat{\Lambda}[\nabla u]^\top - D(u\cdot\nabla\hat{\Lambda})\Big) + \frac{\mu}{2}\int_{\mathcal{F}(0)}|\nabla u_0|^2.
\end{aligned} \tag{103}
$$

We split the right-hand side of this inequality into six integrals denoted $I_1,\ldots,I_6$ that we bound independently. Applying interpolation inequalities for estimating the L^4-norm, we have

$$
\begin{aligned}
I_1 &:= \int_{\mathcal{Q}_t}\rho_f|u\cdot\nabla u|^2 \\
&\leqq C_0(t)\int_0^t\Big(\|u\|^2_{L^4_\sharp(\mathcal{F}(s))}\|\nabla u\|^2_{L^4_\sharp(\mathcal{F}(s))}\Big) \\
&\leqq C_0(t)\int_0^t\Big(\|u\|_{L^2_\sharp(\mathcal{F}(s))}\|\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))}\|u\|_{H^2_\sharp(\mathcal{F}(s))}\Big).
\end{aligned}
$$

Next we use the elliptic estimates derived in Section 3.2 to bound $\|u\|_{H^2(\mathcal{F}(s))}$.

$$\|u\|_{H^2_\sharp(\mathcal{F}(s))} \leqq K^s(R_t)\left(\|\partial_t u + u \cdot \nabla u\|_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_t h\|_{H^{\frac{3}{2}}_\sharp(0,L)}\right). \tag{104}$$

Following Remark 5, the function of time $t \mapsto K^s(R_t)$ is (non-decreasing) and bounded on all bounded subintervals of $[0, T^*)$. Thus, I_1 can be bounded by:

$$C_0(t) \int_0^t \|u\|_{L^2_\sharp(\mathcal{F}(s))} \|\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} \cdots$$
$$\cdots \left(\|\partial_t u + u \cdot \nabla u\|_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_t h\|_{H^{\frac{3}{2}}_\sharp(0,L)}\right).$$

Recalling the energy estimate proven in Proposition 2 we obtain, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} I_1 \leqq C_0(t) &\int_0^t \|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} \\ &+ \varepsilon \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)}\right). \end{aligned} \tag{105}$$

Here and in what follows, we skip the dependance of $C_0(t)$ on ε as this parameter will be fixed later on. The second term reads:

$$I_2 := \int_{\mathcal{Q}_t} \rho_f (\partial_t u + u \cdot \nabla u) \cdot (\hat{\Lambda} \cdot \nabla u),$$

which we estimate as follows:

$$\begin{aligned} |I_2| &\leqq \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|_{L^4_\sharp(\mathcal{F}(s))} \|\nabla u\|_{L^4_\sharp(\mathcal{F}(s))}\right) \\ &\leqq C_0(t) \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|\nabla u\|^{\frac{1}{2}}_{L^2_\sharp(\mathcal{F}(s))} \cdots \right. \\ &\qquad \left. \cdots \|u\|^{\frac{1}{2}}_{H^2_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|^{\frac{1}{2}}_{L^2_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|^{\frac{1}{2}}_{H^1_\sharp(\mathcal{F}(s))}\right). \end{aligned}$$

Thanks to estimates (95), (96) being satisfied by $\hat{\Lambda}$ and using Proposition 2 which enables us to bound $\|u\|_{L^2_\sharp(\mathcal{F}(s))}$ and $\|\partial_t h\|_{L^2_\sharp(0,L)}$, and the elliptic estimate (104), we get, for arbitrary $\varepsilon > 0$ to be fixed later on:

$$\begin{aligned} |I_2| \leqq C_0(t) &\int_0^t \left(\|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{tx} h\|^4_{L^2_\sharp(0,L)}\right) \\ &+ \varepsilon \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)}\right). \end{aligned} \tag{106}$$

The third term reads:

$$I_3 := \int_{\mathcal{Q}_t} \rho_f (\partial_t u + u \cdot \nabla u) \cdot (u \cdot \nabla \hat{\Lambda}).$$

We estimate it as follows:

$$\begin{aligned}|I_3| &\leqq \int_0^t \left(\|\partial_t u + u\cdot\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|u\|_{L^4_\sharp(\mathcal{F}(s))} \|\nabla \hat{\Lambda}\|_{L^4_\sharp(\mathcal{F}(s))} \right) \\ &\leqq C_0(t) \int_0^t \Bigg(\|\partial_t u + u\cdot\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|u\|^{\frac12}_{L^2_\sharp(\mathcal{F}(s))} \cdots \\ &\qquad \cdots \|\nabla u\|^{\frac12}_{L^2_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|^{\frac12}_{H^1_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|^{\frac12}_{H^2_\sharp(\mathcal{F}(s))} \Bigg).\end{aligned}$$

Applying estimates (96) and (97) satisfied by $\hat{\Lambda}$, together with interpolation inequalities and Proposition 2, we obtain

$$\begin{aligned}|I_3| \leqq C_0(t) \int_0^t \Bigg(&\|\partial_t u + u\cdot\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|\nabla u\|^{\frac12}_{L^2_\sharp(\mathcal{F}(s))} \cdots \\ &\cdots \|\partial_{tx} h\|^{\frac12}_{L^2_\sharp(0,L)} \|\partial_{txx} h\|^{\frac12}_{L^2_\sharp(0,L)} \Bigg).\end{aligned}$$

Hence, for arbitrary $\varepsilon > 0$, there holds:

$$\begin{aligned}|I_3| \leqq\ & C_0(t) \int_0^t \left(\|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{tx} h\|^4_{L^2_\sharp(0,L)} \right) \\ &+\varepsilon \int_0^t \left(\|\partial_t u + u\cdot\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)} \right). \qquad (107)\end{aligned}$$

We proceed with

$$I_4 := 2\mu \int_{\mathcal{Q}_t} D(u) : ([\nabla\hat{\Lambda}]^\top \nabla u + \nabla\hat{\Lambda}[\nabla u]^\top),$$

that we bound as follows (C is a constant depending only on μ):

$$\begin{aligned}|I_4| &\leqq C \int_0^t \left(\|\nabla u\|^2_{L^4_\sharp(\mathcal{F}(s))} \|\nabla\hat{\Lambda}\|_{L^2_\sharp(\mathcal{F}(s))} \right) \\ &\leqq C_0(t) \int_0^t \left(\|\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|u\|_{H^2_\sharp(\mathcal{F}(s))} \|\hat{\Lambda}\|_{H^1_\sharp(\mathcal{F}(s))} \right).\end{aligned}$$

Similarly to above, using (96) and (104), we obtain, for arbitrary $\varepsilon > 0$:

$$\begin{aligned}|I_4| \leqq\ & C_0(t) \int_0^t \left(\|\partial_{tx} h\|^4_{L^2_\sharp(0,L)} + \|\nabla u\|^4_{L^2(\mathcal{F}(s))} \right) \\ &+\varepsilon \int_0^t \left(\|\partial_t u + u\cdot\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)} \right). \qquad (108)\end{aligned}$$

Finally, the fifth term is defined by

$$I_5 := 2\mu \int_{\mathcal{Q}_t} D(u) : D(u\cdot\nabla\hat{\Lambda}).$$

Expanding $D(u \cdot \nabla \hat{\Lambda})$, we obtain

$$|I_5| \leqq C \left(\int_{\mathcal{Q}_t} |\nabla u|^2 |\nabla \hat{\Lambda}| + \int_{\mathcal{Q}_t} |\nabla u||u||\nabla^2 \hat{\Lambda}| \right).$$

The first term on the right-hand side is bounded as I_4 (see (108)). As for the second term of the right-hand side, we have

$$\begin{aligned} \int_{\mathcal{Q}_t} |\nabla u||u||\nabla^2 \hat{\Lambda}| &\leqq C_0(t) \int_0^t \Big(\|\hat{\Lambda}\|_{H^2_\sharp(\mathcal{F}(s))} \|u\|^{\frac{1}{2}}_{L^2_\sharp(\mathcal{F}(s))} \cdots \\ &\quad \cdots \|\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|u\|^{\frac{1}{2}}_{H^2_\sharp(\mathcal{F}(s))} \Big) \\ &\leqq C_0(t) \int_0^t \left(\|\partial_{txx} h\|_{L^2_\sharp(0,L)} \|\nabla u\|_{L^2_\sharp(\mathcal{F}(s))} \|u\|^{\frac{1}{2}}_{H^2_\sharp(\mathcal{F}(s))} \right). \end{aligned}$$

Consequently, we obtain:

$$\begin{aligned} \int_{\mathcal{Q}_t} |\nabla u||u||\nabla^2 \hat{\Lambda}| &\leqq C_0(t) \int_0^t \|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} \\ &\quad + \varepsilon \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)} \right). \end{aligned}$$

Finally, for arbitrary small $\varepsilon > 0$, we have:

$$\begin{aligned} |I_5| &\leqq C_0(t) \int_0^t \left(\|\partial_{tx} h\|^4_{L^2_\sharp(0,L)} + \|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} \right) \\ &\quad + \varepsilon \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)} \right). \end{aligned} \tag{109}$$

Introducing (105)–(106)–(107)–(108)–(109) into (103), we obtain that, for arbitrary $\varepsilon > 0$, there holds:

$$\begin{aligned} &\frac{\mu}{2} \int_{\mathcal{F}(t)} |\nabla u|^2 + \frac{1}{4} \int_{\mathcal{Q}_t} \rho_f |\partial_t u + u \cdot \nabla u|^2 + \int_0^t \int_0^L \phi(u, p, h) \partial_{tt} h \\ &\leqq C_0(t) \int_0^t \left(\|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{tx} h\|^4_{L^2_\sharp(0,L)} \right) \\ &\quad + \varepsilon \int_0^t \left(\|\partial_t u + u \cdot \nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{txx} h\|^2_{L^2_\sharp(0,L)} \right) + \frac{\mu}{2} \int_{\mathcal{F}(0)} |\nabla u_0|^2. \end{aligned} \tag{110}$$

We now take care of the elastic part; we apply (101) of Lemma 2 for $b = h$ and rewrite (99) as

$$\begin{aligned} &\int_0^t \int_0^L \rho_s |\partial_{tt} h|^2 + \int_0^L \left(\frac{\gamma}{2} |\partial_{tx} h|^2 - \beta \partial_t h \partial_{xx} h - \alpha \partial_{tx} h \partial_{xxx} h \right) \\ &\quad - \beta \int_0^t \int_0^L |\partial_{tx} h|^2 - \alpha \int_0^t \int_0^L |\partial_{txx} h|^2 = \int_0^t \int_0^L \phi(u, p, h) \partial_{tt} h + C_0. \end{aligned} \tag{111}$$

We add (110) and (111), restricted to $\varepsilon < \min(\rho_f/8, 1)$ and bound all possible terms by energy estimate (Proposition 2). This yields:

$$\frac{\mu}{2}\int_{\mathcal{F}(t)} |\nabla u|^2 + \frac{1}{8}\int_{\mathcal{Q}_t} \rho_f |\partial_t u + u\cdot\nabla u|^2 + \int_0^t\int_0^L \rho_s|\partial_{tt}h|^2 + Rem(t)$$
$$\leqq C_0(t)\int_0^t \left(\|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{tx}h\|^4_{L^2_\sharp(0,L)}\right) + (\alpha+1)\int_0^t\int_0^L |\partial_{txx}h|^2 + C_0, \tag{112}$$

where, thanks to Proposition 2, for any $t \in [0, T]$:

$$\mathrm{Rem}(t) = \int_0^L \left(\frac{\gamma}{2}|\partial_{tx}h|^2 - \beta\partial_t h\partial_{xx}h - \alpha\partial_{tx}h\partial_{xxx}h\right)$$

satisfies:

$$\mathrm{Rem}(t) \geqq -C_0 - \frac{\alpha}{2}\left(\|\partial_{xxx}h(\cdot,t)\|^2_{L^2(0,L)} + \|\partial_{tx}h(\cdot,t)\|^2_{L^2(0,L)}\right).$$

Finally, we get:

$$\frac{\mu}{2}\int_{\mathcal{F}(t)} |\nabla u|^2 - \frac{\alpha}{2}\int_0^L \left(|\partial_{xxx}h|^2 + |\partial_{tx}h|^2\right)$$
$$+\frac{1}{8}\int_{\mathcal{Q}_t} \rho_f|\partial_t u + u\cdot\nabla u|^2 + \int_0^t\int_0^L \rho_s|\partial_{tt}h|^2$$
$$\leqq C_0(t)\int_0^t\left(\|\nabla u\|^4_{L^2_\sharp(\mathcal{F}(s))} + \|\partial_{tx}h\|^4_{L^2_\sharp(0,L)}\right) + (\alpha+1)\int_0^t\int_0^L |\partial_{txx}h|^2 + C_0. \tag{113}$$

To go further, we need to obtain an estimate for:

$$\int_0^t\int_0^L |\partial_{txx}h|^2, \qquad \int_0^L \left(|\partial_{xxx}h(x,t)|^2 + |\partial_{tx}h(x,t)|^2\right)\mathrm{d}x.$$

The next step is thus an independent regularity estimate for the beam equation.

Step 2: Regularity estimate for structure equation – First, $-\partial_{txx}h \in L^2_\sharp((0,L)\times(0,T))$ so it is a suitable multiplier for (3). We obtain after integration:

$$-\int_0^t\int_0^L \rho_s\partial_{tt}h\partial_{txx}h + \beta\int_0^t\int_0^L \partial_{xx}h\partial_{txx}h$$
$$-\alpha\int_0^t\int_0^L \partial_{xxxx}h\partial_{txx}h + \gamma\int_0^t\int_0^L |\partial_{txx}h|^2$$
$$= -\int_0^t\int_0^L \phi(u,p,h)\partial_{txx}h.$$

We can further integrate by parts

$$\frac{1}{2}\int_0^L \Big[\rho_s|\partial_{tx}h|^2 + \alpha|\partial_{xxx}h|^2 + \beta|\partial_{xx}h|^2\Big] \\ +\gamma\int_0^t\int_0^L |\partial_{txx}h|^2 = -\int_0^t\int_0^L \phi(u,p,h)\partial_{txx}h \\ +\frac{1}{2}\int_0^L \Big[\rho_s|\partial_x\dot{h}_0|^2 + \alpha|\partial_{xxx}h_0|^2 + \beta|\partial_{xx}h_0|^2\Big]. \tag{114}$$

Note that all terms make sense due to the regularity of the strong solution h. We control the first term of the right-hand side with the help of Corollary 2, recalling that constant $K^b(R_t)$ defines a function of time that remains bounded on bounded subintervals of $[0, T^*)$. Consequently, we obtain:

$$\left|\int_0^L \phi(u,p,h)\partial_{txx}h\right| \leqq \|\phi(u,p,h)\|_{H_\sharp^{\frac{1}{2}}(0,L)}\|\partial_{txx}h\|_{H_\sharp^{-\frac{1}{2}}(0,L)} \\ \leqq C_0(t)\left(\|\mu\Delta u - \nabla p\|_{L^2_\sharp(\mathcal{F}(t))} + \|\partial_t h\|_{H_\sharp^{\frac{3}{2}}(0,L)}\right)\|\partial_t h\|_{H_\sharp^{\frac{3}{2}}(0,L)}.$$

Since $\mu\Delta u - \nabla p = \partial_t u + u\cdot\nabla u$ we then have

$$\left|\int_0^L \phi(u,p,h)\partial_{txx}h\right| \\ \leqq C_0(t)\left(\|\partial_t u + u\cdot\nabla u\|_{L^2_\sharp(\mathcal{F}(t))} + \|\partial_t h\|_{H_\sharp^{\frac{3}{2}}(0,L)}\right)\|\partial_t h\|_{H_\sharp^{\frac{3}{2}}(0,L)}.$$

Moreover, we apply the following interpolation inequality:

$$\|\partial_t h(\cdot,t)\|_{H_\sharp^{\frac{3}{2}}(0,L)} \leqq C\|\partial_{tx}h(\cdot,t)\|^{\frac{1}{2}}_{L^2_\sharp(0,L)}\|\partial_{txx}h(\cdot,t)\|^{\frac{1}{2}}_{L^2_\sharp(0,L)}.$$

This yields:

$$\left|\int_0^L \phi(u,p,h)\partial_{txx}h\right| \leqq C_0(t)\|\partial_{tx}h\|^2_{L^2_\sharp(0,L)} \\ +\varepsilon\left(\|\partial_t u + u\cdot\nabla u\|^2_{L^2_\sharp(\mathcal{F}(t))} + \|\partial_{txx}h\|^2_{L^2_\sharp(0,L)}\right). \tag{115}$$

Introducing (115) into (114), we obtain that, for arbitrary $\varepsilon > 0$ sufficiently small, there holds:

$$\frac{1}{2}\int_0^L \Big[\rho_s|\partial_{tx}h|^2 + \alpha|\partial_{xxx}h|^2\Big] + \frac{\gamma}{2}\int_0^t\int_0^L |\partial_{txx}h|^2 \\ \leqq C_0(t)\int_0^t \|\partial_{tx}h\|^2_{L^2_\sharp(0,L)} + \varepsilon\int_0^t \|\partial_t u + u\cdot\nabla u\|^2_{L^2_\sharp(\mathcal{F}(s))} + C_0. \tag{116}$$

Step 3 : Conclusion. We multiply (116) by a sufficiently large constant, typically,

$$\max\left(\frac{\alpha}{\rho_s}, 4, \frac{4(\alpha+1)}{\gamma}\right),$$

and adding (113), we obtain, choosing ε sufficiently small:

$$\begin{aligned}
&\frac{\mu}{2}\int_{\mathcal{F}(t)} |\nabla u|^2 + \int_{\mathcal{Q}_t} |\partial_t u + u\cdot\nabla u|^2 \\
&\quad +\delta\left(\frac{1}{2}\int_0^L \left[\rho_s|\partial_{tx}h|^2 + \alpha|\partial_{xxx}h|^2\right] + \frac{\gamma}{2}\int_0^t\int_0^L |\partial_{txx}h|^2\right) \\
&\leqq C_0(t)\int_0^t \left(1 + \|\nabla u\|^2_{L^2(\mathcal{F}(s))} + \|\partial_{tx}h\|^2_{L^2_\sharp(0,L)}\right)^2 + C_0,
\end{aligned} \tag{117}$$

where $\delta > 0$. Setting

$$\mathcal{E}_{reg} := \frac{\mu}{2}\int_{\mathcal{F}(t)} |\nabla u|^2 + \delta\left(\frac{1}{2}\int_0^L \left[\rho_s|\partial_{tx}h|^2 + \alpha|\partial_{xxx}h|^2\right]\right),$$

we then have for all $t \in [0, T]$:

$$\|u(\cdot,t)\|^2_{H^1(\mathcal{F}(t))} + \|\partial_t h(\cdot,t)\|^2_{H^1(0,L)} + \|h(\cdot,t)\|_{H^3(0,L)} \leqq C_0 + C\mathcal{E}_{\mathrm{reg}}(t),$$

with a constant C depending only on μ, ρ_s, α and δ, and

$$\begin{aligned}
\mathcal{E}_{\mathrm{reg}}(t) \leqq C_0(t)\int_0^t \Bigg(\left(1 + \|\nabla u\|^2_{L^2(\mathcal{F}(s))} + \|\partial_{tx}h\|^2_{L^2_\sharp(0,L)}\right)\ldots \\
\ldots\left(\mathcal{E}_{\mathrm{reg}}(s) + 1\right) + C_0\Bigg).
\end{aligned}$$

Proposition 2 implies then that:

$$\int_0^t \left(1 + \|\nabla u\|^2_{L^2(\mathcal{F}(t))} + \|\partial_{tx}h\|^2_{L^2_\sharp(0,L)}\right) \leqq C_0(t).$$

We thus complete the proof of Proposition 4 by applying a Gronwall lemma and remembering that $t \mapsto C_0(t)$ is bounded on any bounded interval.

Appendix A: Technical Details for the No-Contact Result

In this appendix we collect technical lemmas that are used throughout Section 4.

A.1 Estimating Positivity of Scalar Functions

We prove first functional inequalities which enable to bound from below a positive function.

Proposition 5. *Let* $\alpha \in (0, 1/2)\backslash\{1/4\}$. *Given a positive function* $\eta \in H^2_\sharp(0, L)$, *there holds:*

$$\int_0^L \frac{|\partial_x \eta|^4}{|\eta|^{4\alpha}} \leqq C\|\partial_{xx}\eta\|^2_{L^2_\sharp(0,L)}\|\eta\|^{2(1-2\alpha)}_{L^\infty_\sharp(0,L)}.$$

Proof. Without restriction, we assume that η is smooth and we set

$$b(x) = [\eta(x)]^{1-\alpha} \quad \forall\, x \in (0, L).$$

A straightforward integration by parts together with the L-periodicity, yields

$$\int_0^L |\partial_x b|^4 + 3\int_0^L b\ \partial_{xx}b\ |\partial_x b|^2 = 0.$$

Replacing b by its value, we obtain:

$$(1-\alpha)^3(1-4\alpha)\int_0^L \frac{|\partial_x \eta|^4}{|\eta|^{4\alpha}} + 3(1-\alpha)^3\int_0^L \left[|\eta|^{(1-2\alpha)}\partial_{xx}\eta\right]\frac{|\partial_x \eta|^2}{|\eta|^{2\alpha}} = 0.$$

When $\alpha \neq 1/4$ this yields

$$\int_0^L \frac{|\partial_x \eta|^4}{|\eta|^{4\alpha}} \leqq C\int_0^L \left[|\eta|^{(1-2\alpha)}\partial_{xx}\eta\right]\frac{|\partial_x \eta|^2}{|\eta|^{2\alpha}}.$$

We conclude by applying Cauchy-Schwarz inequality, which yields, since $\alpha < 1/2$,

$$\begin{aligned}\int_0^L \frac{|\partial_x \eta|^4}{|\eta|^{4\alpha}} &\leqq C\int_0^L |\eta|^{2(1-2\alpha)}|\partial_{xx}\eta|^2 \\ &\leqq C\|\eta\|^{2(1-2\alpha)}_{L^\infty_\sharp(0,L)}\|\partial_{xx}\eta\|^2_{L^2_\sharp(0,L)}.\end{aligned}$$

□

Proposition 6. *Given a positive function* $\eta \in H^3_\sharp(0, L)$, *the following pointwise estimates hold true:*

$$|\partial_x \eta(x)|^2 \leqq C\|\partial_{xx}\eta\|^{\frac{1}{2}}_{L^2_\sharp(0,L)}\|\partial_{xxx}\eta\|^{\frac{1}{2}}_{L^2_\sharp(0,L)}\,|\eta(x)|, \quad \forall\, x \in (0, L). \tag{118}$$

$$|\partial_x \eta(x)|^2 \leqq C\|\partial_{xxx}\eta\|_{L^2_\sharp(0,L)}\,|\eta(x)|, \quad \forall\, x \in (0, L). \tag{119}$$

Proof. Let us denote $z \in H^3_\sharp(0, L)$ a positive function.

$$|\partial_x z(x)| \leqq \||\partial_x z|^2 + z\partial_{xx}z\|^{\frac{1}{2}}_{L^\infty_\sharp(0,L)}, \quad \forall\, x \in (0, L). \tag{120}$$

Indeed, as z is L-periodic, $\partial_x z$ reaches its maximal and minimal values. Then, if $\partial_x z$ is maximal at $x_0 \in (0, L)$, then $\partial_{xx}z(x_0) = 0$ and consequently

$$|\partial_x z(x_0)| \leqq \sqrt{|\partial_x z(x_0)|^2 + z\partial_{xx}z(x_0)} \leqq \||\partial_x z|^2 + z\partial_{xx}z\|^{\frac{1}{2}}_{L^\infty_\sharp(0,L)}.$$

We get a similar inequality where $\partial_x z$ is minimal and obtain (120).

We now apply the previous estimate with

$$z(x) = \sqrt{\eta(x)}, \quad \forall\, x \in (0, L).$$

Of course z is a positive L-periodic function which belongs to $H^3_\sharp(0, L)$ so it satisfies (120). Replacing z by $\sqrt{\eta}$ leads to

$$|\partial_x \eta(x)| \leqq C\sqrt{\|\partial_{xx}\eta\|_{L^\infty_\sharp(0,L)}\, \eta(x)}, \quad \forall\, x \in (0, L).$$

Thanks to the continuous embedding $H^1_\sharp(0, L) \cap L^2_{\sharp,0}(0, L)$, in $L^\infty_\sharp(0, L)$, we obtain (119) and the interpolation inequality

$$\|\partial_{xx}\eta\|_{L^\infty_\sharp(0,L)} \leqq \|\partial_{xx}\eta\|^{1/2}_{L^2_\sharp(0,L)} \|\partial_{xxx}\eta\|^{1/2}_{L^2_\sharp(0,L)}$$

implies that (118) is satisfied. □

Proposition 7. *There exists a continuous function $D_{\min} : (0, \infty) \times \mathbb{R}^+ \to (0, \infty)$ such that, given a positive function $\eta \in H^2_\sharp(0, L)$, there holds*

$$\|\eta^{-1}\|_{L^\infty_\sharp(0,L)} \leqq D_{\min}\left(\|\eta^{-1}\|_{L^1_\sharp(0,L)}, \|\eta\|_{H^2_\sharp(0,L)}\right). \tag{121}$$

Proof. Without further restriction, we assume η to be smooth. We note that $\|\eta^{-1}\|_{L^\infty_\sharp(0,L)}$ is achieved for some $x_0 \in [0, L]$. Then, as $\partial_x \eta(x_0) = 0$ we have:

$$\eta(x) = \eta(x_0) + \int_{x_0}^{x} (s - x_0)\partial_{xx}\eta(s)\,\mathrm{d}s,$$

At this point, we consider two cases. First, if $\|\partial_{xx}\eta\|_{L^2_\sharp(0,L)} = 0$, we obtain

$$\|\eta^{-1}\|_{L^1_\sharp(0,L)} = L\|\eta^{-1}\|_{L^\infty_\sharp(0,L)}. \tag{122}$$

Second, if $\|\partial_{xx}\eta\|_{L^2_\sharp(0,L)} > 0$, we have the bound:

$$\eta(x) \leqq \eta(x_0) + \frac{1}{\sqrt{3}}\|\eta\|_{H^2_\sharp(0,L)}|x - x_0|^{\frac{3}{2}}, \quad \forall\, x \in (x_0 - L, x_0 + L).$$

For simplicity, let denote $\eta_m := \eta(x_0)$ and $M = \|\eta\|_{H^2_\sharp(0,L)}/\sqrt{3}$. We have then

$$\|\eta^{-1}\|_{L^1_\sharp(0,L)} = \int_{x_0}^{x_0+L} \frac{\mathrm{d}s}{\eta(s)} \geqq \int_0^L \frac{\mathrm{d}s}{\eta_m + M|s|^{\frac{3}{2}}} \geqq \frac{1}{M^{\frac{2}{3}}\eta_m^{\frac{1}{3}}}\phi\left(\left[\frac{M}{\eta_m}\right]^{\frac{2}{3}}\right),$$

where we applied a straightforward change of variable to introduce

$$\phi(\sigma) := \int_0^{L\sigma} \frac{\mathrm{d}z}{(1 + |z|^{\frac{3}{2}})}.$$

We remark that $\phi : \mathbb{R}^+ \to \mathbb{R}$ is an increasing continuous function such that

$$\phi(\sigma) \sim \begin{cases} L\sigma & \text{for } \sigma << 1, \\ \displaystyle\int_{\mathbb{R}^+} \frac{\mathrm{d}z}{1+|z|^{\frac{3}{2}}} & \text{for } \sigma >> 1. \end{cases}$$

Furthermore, applying that $\|\eta^{-1}\|_{L^1_\sharp(0,L)} \leqq L\|\eta^{-1}\|_{L^\infty_\sharp(0,L)} = L\eta_m^{-1}$, we obtain

$$\|\eta^{-1}\|_{L^1_\sharp(0,L)} \geqq \frac{1}{M^{\frac{2}{3}}\eta_m^{\frac{1}{3}}}\phi\left(\left[\frac{M\|\eta^{-1}\|_{L^1_\sharp(0,L)}}{L}\right]^{\frac{2}{3}}\right).$$

Finally, if $\|\eta\|_{H^2_\sharp(0,L)} > 0$, we have:

$$\begin{aligned} &\|\eta^{-1}\|_{L^\infty_\sharp(0,L)} \\ &\leqq \|\eta\|^2_{H^2_\sharp(0,L)}\|\eta^{-1}\|^3_{L^1_\sharp(0,L)}\left[\phi\left(\left[\frac{\|\eta\|_{H^2_\sharp(0,L)}\|\eta^{-1}\|_{L^1_\sharp(0,L)}}{L}\right]^{\frac{2}{3}}\right)\right]^{-3}. \end{aligned} \tag{123}$$

The above expansion of $\phi(\sigma)$ for small values of σ yields that, when $\|\eta\|_{H^2_\sharp(0,L)} << 1$ with $\|\eta^{-1}\|_{L^1_\sharp(0,L)}$ bounded, there holds

$$\begin{aligned} &\|\eta\|^2_{H^2_\sharp(0,L)}\|\eta^{-1}\|^3_{L^1_\sharp(0,L)}\left[\phi\left(\left[\frac{\|\eta; H^2_\sharp(0,L)\|\|\eta^{-1}\|_{L^1_\sharp(0,L)}}{L}\right]^{\frac{2}{3}}\right)\right]^{-3} \\ &\sim \frac{1}{L}\|\eta^{-1}\|_{L^1_\sharp(0,L)}. \end{aligned}$$

Hence, we set

$$D_{\min}(\alpha,\beta) = \frac{\alpha}{L}, \text{ if } \beta = 0, \qquad D_{\min}(\alpha,\beta) = \frac{\beta^2\alpha^3}{9}\left[\phi\left(\left[\frac{\beta\alpha}{L}\right]^{\frac{2}{3}}\right)\right]^{-3}.$$

Because of the previous arguments, this is a continuous function on $(0,\infty)\times[0,\infty)$ which satisfies (121). □

A.2 Estimates on the Stream-Function ψ

In this appendix, we gather technical estimates regarding the stream-function

$$\psi(x,y,t) = \partial_x h(x,t)\chi_0\left(\frac{y}{h(x,t)}\right) \quad \forall\,(x,y,t) \in \mathcal{Q}_T,$$

where $h \in H^2(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^4_\sharp(0,L))$ is given and remains strictly positive on $(0,T)$. We recall that

$$\chi_0(z) = z^2(3-2z) \quad \forall\, z \in (0,1),$$

and that $\mathcal{F}(t)$ stands for the fluid domain at time t and $\mathcal{Q}_T$ is the time-space fluid domain. With these notations, we prove:

Proposition 8. *There exists a universal constant $C < \infty$ for which:*

– *for all $t \in (0,T)$ for all $(x,y) \in \mathcal{F}(t)$,*

$$|\nabla\psi(x,y,t)| \leqq C\left[|\partial_{xx}h(x,t)| + \frac{|\partial_x h(x,t)|}{h(x,t)} + \frac{|\partial_x h(x,t)|^2}{h(x,t)}\right]; \tag{124}$$

– *for all $t \in (0,T)$,*

$$\|\partial_y\psi(\cdot,t)\|_{L^2(\mathcal{F}(t))} \leqq C\|h\|^{\frac{1}{4}}_{L^\infty_\sharp(0,L)}\|\partial_{xx}h\|^{\frac{1}{2}}_{L^2_\sharp(0,L)}\left[\int_0^L \frac{1}{h}\right]^{\frac{1}{4}}, \tag{125}$$

$$\|\partial_x\psi(\cdot,t)\|_{L^2(\mathcal{F}(t))} \leqq C\|h\|^{\frac{1}{2}}_{L^\infty_\sharp(0,L)}\|\partial_{xx}h\|_{L^2_\sharp(0,L)}; \tag{126}$$

– *on the whole time–space domain,*

$$\|\partial_t\psi\|_{L^2(\mathcal{Q}_T)} \leqq C\bigg[\int_0^T \Big(\|h\|_{L^\infty_\sharp(0,L)}\|\partial_{xt}h\|^2_{L^2_\sharp(0,L)} + \|\partial_t h\|^2_{L^2_\sharp(0,L)}\|\partial_{xxx}h\|_{L^2_\sharp(0,L)}\Big)\bigg]^{\frac{1}{2}}, \tag{127}$$

$$\|\partial_{xx}\psi\|_{L^2(\mathcal{Q}_T)} \leqq C\bigg[\int_0^T \Big(\|h\|_{L^\infty_\sharp(0,L)}\|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)} + \|\partial_{xx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)}\|\partial_{xxx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)}\Big)\bigg]^{\frac{1}{2}}. \tag{128}$$

Proof. We start by the time-dependant estimate. We compute the partial derivative of the stream-function with respect to the space variables, for any $(x,y,t) \in \mathcal{Q}_T$:

$$\begin{cases} \partial_x\psi(x,y,t) = \partial_{xx}h(x,t)\chi_0\left(\dfrac{y}{h(x,t)}\right) - \dfrac{|\partial_x h(x,t)|^2 y}{|h(x,t)|^2}\chi_0'\left(\dfrac{y}{h(x,t)}\right), \\ \partial_y\psi(x,y,t) = \dfrac{\partial_x h(x,t)}{h(x,t)}\chi_0'\left(\dfrac{y}{h(x,t)}\right). \end{cases}$$

Thus (124) is satisfied. Furthermore, we have, for all $t \in (0,T)$, recalling that $\chi_0 \in C^\infty([0,1])$,

$$\begin{aligned} \|\partial_y\psi\|^2_{L^2(\mathcal{F}(t))} &= \int_0^L\int_0^{h(x,t)} \frac{|\partial_x h(x,t)|^2}{|h(x,t)|^2}\left|\chi_0'\left(\frac{y}{h(x,t)}\right)\right|^2 \mathrm{d}y\mathrm{d}x \\ &= \int_0^L\int_0^1 \frac{|\partial_x h(x,t)|^2}{h(x,t)}\left|\chi_0'(z)\right|^2 \mathrm{d}y\mathrm{d}z. \end{aligned}$$

Then applying Proposition 5 with $\alpha = 3/8$, we have:

$$\begin{aligned}\|\partial_y \psi\|^2_{L^2(\mathcal{F}(t))} &\leqq C \left[\int_0^L \frac{|\partial_x h|^4}{h}\right]^{\frac{1}{2}} \left[\int_0^L \frac{1}{h}\right]^{\frac{1}{2}} \\ &\leqq C\ \|h\|^{\frac{1}{4}}_{L^\infty_\sharp(0,L)} \left[\int_0^L \frac{|\partial_x h|^4}{|h|^{\frac{3}{2}}}\right]^{\frac{1}{2}} \left[\int_0^L \frac{1}{h}\right]^{\frac{1}{2}} \\ &\leqq C\ \|h\|^{\frac{1}{2}}_{L^\infty_\sharp(0,L)} \|\partial_{xx} h\|_{L^2_\sharp(0,L)} \left[\int_0^L \frac{1}{h}\right]^{\frac{1}{2}}.\end{aligned}$$

With similar arguments, we easily obtain

$$\|\partial_x \psi\|^2_{L^2(\mathcal{F}(t))} \leqq C \left[\|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xx} h\|^2_{L^2_\sharp(0,L)} + \int_0^L \frac{|\partial_x h|^4}{h}\right].$$

Applying again Proposition 5 with $\alpha = 3/8$, the following estimate holds true in a way similar to as above:

$$\|\partial_x \psi\|^2_{L^2(\mathcal{F}(t))} \leqq C \|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xx} h\|^2_{L^2_\sharp(0,L)}.$$

This ends the proof of (125) and (126).

We now take care of the two other estimates. A simple calculation gives

$$\begin{aligned}\partial_t \psi(x, y, t) &= \partial_{xt} h(x,t) \chi_0 \left(\frac{y}{h(x,t)}\right) \\ &\quad - \frac{\partial_x h(x,t) \partial_t h(x,t) y}{|h(x,t)|^2} \chi_0' \left(\frac{y}{h(x,t)}\right), \quad \forall (x,y,t) \in \mathcal{Q}_T.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\|\partial_t \psi\|^2_{L^2(\mathcal{Q}_T)} \\ &\quad \leqq C \int_0^T \left(\|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xt} h\|^2_{L^2_\sharp(0,L)} + \int_0^L \frac{|\partial_x h|^2 |\partial_t h|^2}{h}\right).\end{aligned}$$

With Proposition 6 we bound the ratio $|\partial_x h|^2/h$ in the right-hand side, and we obtain

$$\begin{aligned}&\|\partial_t \psi\|^2_{L^2(\mathcal{Q}_T)} \\ &\quad \leqq C \int_0^T \left(\|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xt} h\|^2_{L^2_\sharp(0,L)} + \|\partial_t h\|^2_{L^2_\sharp(0,L)} \|\partial_{xxx} h\|_{L^2_\sharp(0,L)}\right).\end{aligned}$$

Finally, we estimate $\partial_{xx}\psi$ which is equal to

$$\begin{aligned}\partial_{xx} \psi(x,y,t) &= \partial_{xxx} h(x,t)\ \chi_0 \left(\frac{y}{h(x,t)}\right) - \frac{3y \partial_{xx} h(x,t) \partial_x h(x,t)}{(h(x,t))^2} \chi_0' \left(\frac{y}{h(x,t)}\right) \\ &\quad + \frac{(\partial_x h(x,t))^3 y^2}{(h(x,t))^4} \chi_0'' \left(\frac{y}{h(x,t)}\right).\end{aligned}$$

Consequently, the following estimate holds true

$$\|\partial_{xx}\psi\|^2_{L^2(Q_T)} \leqq C\int_0^T \left[\|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)} + \int_0^L \left(\frac{|\partial_{xx}h|^2|\partial_x h|^2}{h} + \frac{|\partial_x h|^6}{|h|^3} \right) \right].$$

By estimates (118) and (119) of Proposition 6, we obtain

$$\|\partial_{xx}\psi\|^2_{L^2(Q_T)} \leqq C\int_0^T \Big[\|h\|_{L^\infty_\sharp(0,L)} \|\partial_{xxx}h\|^2_{L^2_\sharp(0,L)} + \|\partial_{xx}h\|^2_{L^2_\sharp(0,L)} \|\partial_{xxx}h\|_{L^2_\sharp(0,L)} + \|\partial_{xx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)} \|\partial_{xxx}h\|^{\frac{3}{2}}_{L^2_\sharp(0,L)} \Big].$$

Finally (128) is satisfied. □

Appendix B: Technical Lemmas for the Regularity Estimates

In this second appendix, we collect technical results that are applied in the proof of Proposition 4. We first construct the lifting velocity-field $\hat{\Lambda}$ and prove then two identities that are central in the proof of our regularity estimates.

B.1 Construction of $\hat{\Lambda}$

We construct a time-frozen operator U_h which satisfies the equivalent properties to the one we require for $\hat{\Lambda}$. We shall then set $\hat{\Lambda} = U_h[\partial_t h]$. The construction of U_h is the content of the following proposition:

Proposition 9. *Let $h \in H^2_\sharp(0, L)$ such that $h^{-1} \in L^\infty_\sharp(0, L)$, there exists a continuous linear mapping $U_h : H^1_\sharp(0, L) \cap L^2_{\sharp,0}(0, L) \to H^1_\sharp(\Omega_h)$ s.t. :*

- *for all $\dot{\eta} \in H^1_\sharp(0, L) \cap L^2_{\sharp,0}(0, L)$ we have*

$$U_h[\dot{\eta}](x, 0) = 0, \quad U_h[\dot{\eta}](x, h(x)) = \dot{\eta}(x)e_2, \quad \forall x \in (0, L);$$
$$\operatorname{div} U_h[\dot{\eta}] = 0 \quad on \quad \Omega_h$$

- *for all $\dot{\eta} \in H^2_\sharp(0, L) \cap L^2_{\sharp,0}(0, L)$ there holds $U_h[\dot{\eta}] \in H^2_\sharp(\Omega_h)$.*

Furthermore we construct U_h such that:

- $\partial_2 U_h[\dot{\eta}](x, h(x)) = 0.$
- *there exists a constant K^l depending increasingly on $\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)}$ for which*

$$\|U_h[\dot{\eta}]\|_{L^2_\sharp(\Omega_h)} \leqq K^l \|\dot{\eta}\|_{L^2(0,L)}, \tag{129}$$

$$\|U_h[\dot{\eta}]\|_{H^1_\sharp(\Omega_h)} \leqq K^l \|\dot{\eta}\|_{H^1(0,L)}, \tag{130}$$

$$\|U_h[\dot{\eta}]\|_{H^2_\sharp(\Omega_h)} \leqq K^l \|\dot{\eta}\|_{H^2(0,L)}, \quad \forall \dot{\eta} \in H^2_\sharp(0, L) \cap L^2_{\sharp,0}(0, L). \tag{131}$$

Proof. We consider $h \in H^2_\sharp(0,L)$ which does not vanish on $(0,L)$ and assume that

$$\|h\|_{H^2_\sharp(0,L)} + \|h^{-1}\|_{L^\infty_\sharp(0,L)} \leqq R_0.$$

We construct U_h in order that (129)–(130)–(131) holds with a constant K^l depending on R_0 only.

Fix $\lambda = 1/(2R_0)$ so that $\min_{x\in[0,L]} h(x) \geqq 2\lambda$. Assume that $\dot\eta \in H^1_\sharp(0,L)$. We define U_h as

$$U_h[\dot\eta] = \begin{cases} \dot\eta e_2, & \text{in } \ \Omega_h \setminus \overline{(0,L)\times(0,\lambda)}, \\ U_\lambda[\dot\eta], & \text{in } \ (0,L)\times(0,\lambda). \end{cases} \tag{132}$$

Note that the restriction of U_h to $\Omega_h \setminus \overline{(0,L)\times(0,\lambda)}$ obviously satisfies continuity estimates similar to (129)–(130)–(131). With this step, we reduce the construction to a rectangular box. In order to change the boundary on which we have to match the data, we define $U_\lambda[\dot\eta]$ by:

$$U_\lambda[\dot\eta](x,y) = \begin{pmatrix} -\frac{1}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} \tilde U[\dot\eta]\left(x, \frac{\lambda - y}{\lambda}\right), \quad \forall (x,y) \in (0,L)\times(0,\lambda).$$

Let us define now $\tilde U[\dot\eta]$. We expand $\dot\eta \in H^2_\sharp(0,L)$ in Fourier series as

$$\dot\eta(x) = \sum_{n\in\mathbb{Z}} \dot\eta_n \exp\left(in\frac{\pi x}{L}\right).$$

By assumption, we have $\dot\eta_0 = 0$. Let $k \in \mathbb{N}$. Then, choosing Q a polynomial function such that:

$$Q(0) = 1 \quad Q'(0) = Q''(0) = 0 \tag{133}$$
$$Q^{(l)}(1) = 0 \quad \forall\, l \leqq k+1. \tag{134}$$

and, for all $n \in \mathbb{Z}$:

$$P_n(z) := Q(\min(|n|z, 1)) \quad \forall\, z \in [0,\infty),$$

we obtain that $P_n \in C^{k+1}([0,1])$. Consequently, we define

$$\tilde U[\dot\eta] = \nabla^\perp \Psi, \quad \text{with } \Psi(x,z) := \sum_{n\in\mathbb{Z}} \frac{L\dot\eta_n}{in\pi} P_n(z) \exp\left(in\frac{\pi x}{L}\right).$$

With this definition we have

$$\tilde U[\dot\eta](x,0) = \dot\eta(x)e_2 \ \text{ and } \ U[\dot\eta](x,1) = 0 \quad \text{for all } x \in (0,L), \tag{135}$$
$$\partial_2 \tilde U[\dot\eta](x,0) = 0, \tag{136}$$
$$\operatorname{div} \tilde U[\dot\eta] = 0 \ \text{ on } \Omega_1. \tag{137}$$

Consequently, thanks to (136), we deduce classically that $U_h[\dot\eta]$ defined by (132) belongs to $H^2(\Omega_h)$. Moreover, for all $(j,l) \in \mathbb{N}^2$ s.t. $\max(j,l) \leqq k+1$, we have, because of the x-orthogonality of the basis $x \mapsto \exp(in\dfrac{\pi x}{L})$, that

$$\begin{aligned}\|\partial_x^j \partial_z^l \Psi\|^2_{L^2_\sharp((0,L)\times(0,1))} &\leqq C \sum_{n\in\mathbb{Z}} \int_0^{\frac{1}{n}} |n|^{2(j+l-1)} |\dot\eta_n|^2 |Q^{(l)}(|n|z)|^2 \mathrm{d}z \\ &\leqq C\left(\sum_{n\in\mathbb{Z}} |n|^{2(j+l)-3}|\dot\eta_n|^2\right) \int_0^1 |Q^{(l)}(z)|^2 \mathrm{d}z \\ &\leqq C\|\dot\eta\|^2_{H_\sharp^{j+l-\frac{3}{2}}(0,L)} \|Q\|^2_{H^l(0,1)}.\end{aligned}$$

Consequently, when $\dot\eta \in H_\sharp^{k-1/2}(0,L)$, we get that $\tilde U[\dot\eta] \in H_\sharp^k(\Omega_1)$ and that the following estimate is satisfied:

$$\|\tilde U[\dot\eta]\|_{H_\sharp^k(\Omega_1)} \leqq \|Q\|_{H^{k+1}(0,1)} \|\dot\eta\|_{H_\sharp^{k-\frac{1}{2}}(0,L)}. \tag{138}$$

Up to the change of variable which depends only on λ and is thus bounded by a constant which depends only on R_0, we obtain that U_λ also satisfies continuity estimates similar to (129)–(130)–(131). This ends the proof. □

Remark 6. An alternative construction for U_h reads:

$$U_h[\dot\eta] = B_h^{-\top} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde U[\dot\eta] \left(x, \frac{h(x)-y}{h(x)}\right),$$

where B_h is defined by (52). With this construction, we may exploit the better continuity estimates for $\dot\eta \mapsto U_h[\dot\eta]$ (gain of 1/2 derivative, see (138)). However, this construction requires more regularity of h than the construction given in Proposition 9.

B.2 Regularity Identities

Lemma 3. *Let $h \in H^2(0,T;L^2(0,L)) \cap L^2(0,T,H^4(0,L))$ such that we have $h^{-1} \in L^\infty((0,L)\times(0,T))$. For any triplet (w,q,b) satisfying the regularity assumptions of Definition 1, namely*

$$b \in H^2(0,T;L^2_\sharp(0,L)) \cap L^2(0,T;H^4_\sharp(0,L)), \tag{139}$$

$$w \in H^1_\sharp(\mathcal{Q}_T), \quad \nabla^2 w \in L^2_\sharp(\mathcal{Q}_T), \quad q \in L^2_\sharp(\mathcal{Q}_T), \quad \nabla q \in L^2_\sharp(\mathcal{Q}_T), \tag{140}$$

and such that $w(x,h(x,t),t) = \partial_t b(x,t)e_2 \in L^2_{\sharp,0}(0,L)$ and $\operatorname{div} w = 0$, the following identities are satisfied:

$$\begin{aligned}\int_{\mathcal{Q}_t} \mathrm{div}\sigma(w,q)\cdot(\partial_t w + \hat\Lambda\cdot\nabla w - w\cdot\nabla\hat\Lambda) &= -\int_0^t\int_0^L \phi(w,q,h)\partial_{tt} b \\ &+\mu\int_{\mathcal{F}(t)} |\nabla w|^2(t) - \mu\int_{\mathcal{F}(0)} |\nabla w|^2(0) \\ &+2\mu\int_{\mathcal{Q}_t} D(w) : \left([\nabla\hat\Lambda]^\top\nabla w + \nabla\hat\Lambda[\nabla w]^\top - D(w\cdot\nabla\hat\Lambda)\right),\end{aligned} \tag{141}$$

and

$$\begin{aligned}
&\alpha \int_0^t \int_0^L \partial_{xxxx} b \, \partial_{tt} b - \beta \int_0^t \int_0^L \partial_{xx} b \, \partial_{tt} b - \gamma \int_0^t \int_0^L \partial_{xxt} b \, \partial_{tt} b \\
&= \int_0^L \frac{\gamma}{2} |\partial_{tx} b(x,t)|^2 \mathrm{d}x - \beta \partial_t b(x,t) \partial_{xx} b(x,t) \mathrm{d}x - \alpha \partial_{tx} b(x,t) \partial_{xxx} b(x,t) \mathrm{d}x \\
&\quad - \beta \int_0^t \int_0^L |\partial_{tx} b|^2 - \alpha \int_0^t \int_0^L |\partial_{txx} b|^2 \\
&\quad - \int_0^L \frac{\gamma}{2} |\partial_{tx} b(x,0)|^2 \mathrm{d}x + \beta \partial_t b(x,0) \partial_{xx} b(x,0) \mathrm{d}x \\
&\quad - \alpha \partial_{tx} b(x,0) \partial_{xxx} b(x,0) \mathrm{d}x. \qquad (142)
\end{aligned}$$

Proof. The second identity, involving b only, is straightforward since applying classical interpolation arguments (see in particular [40, Theorem 3.1, p. 19]) the regularity (139) of b implies that

$$b \in C^0((0,T); H^3_\sharp(0,L)) \cap H^1(0,T; H^2_\sharp(0,L)) \cap C^1((0,T); H^1_\sharp(0,L)). \qquad (143)$$

We now explain how one can obtain the first identity. Assuming that (w, b) are regular enough we have

$$\begin{aligned}
&\int_{\mathcal{Q}_t} \mathrm{div}\sigma(w,q) \cdot (\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda}) \\
&= \int_0^t \int_{y=h(x,s)} \sigma(w,q) n \cdot (\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda}) \\
&\quad - \int_0^t \int_{\mathcal{F}(s)} \sigma(w,q) : \nabla(\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda}).
\end{aligned}$$

However by the assumption satisfied by (w, b) on the boundary $y = h(x,t)$ and by the construction of $\hat{\Lambda}$

$$\begin{aligned}
w \cdot \nabla \hat{\Lambda} = w_2 \partial_2 \hat{\Lambda} &= 0, \\
\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda} &= \partial_{tt} b e_2,
\end{aligned} \qquad \text{on } \; y = h(x,t).$$

Combining this with the definition (4) of ϕ, we have

$$\int_0^t \int_{y=h(x,s)} \sigma(w,q) n \cdot (\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda}) = -\int_0^t \int_0^L \phi(w,q,h) \partial_{tt} b.$$

On the other hand, we have by construction since $\mathrm{div}\, w = \mathrm{div}\, \hat{\Lambda} = 0$

$$\mathrm{div} \left(\partial_t w + \hat{\Lambda} \cdot \nabla w - w \cdot \nabla \hat{\Lambda} \right) = 0.$$

Consequently, recalling that $\mathcal{F}(t)$ moves along the characteristics of $\hat{\Lambda}$, we obtain the following equality:

$$
\begin{aligned}
&\int_{\mathcal{Q}_t} \sigma(w,q) : \nabla(\partial_t w + \hat{\Lambda}\cdot\nabla w - w\cdot\nabla\hat{\Lambda}) \\
&= 2\mu \int_{\mathcal{Q}_t} D(w) : D(\partial_t w + \hat{\Lambda}\cdot\nabla w - w\cdot\nabla\hat{\Lambda})) \\
&= \mu \int_{\mathcal{F}(t)} |Dw|^2 - \mu \int_{\mathcal{F}(0)} |Dw|^2(0) \\
&\quad + 2\mu \int_{\mathcal{Q}_t} D(w) : \Big([\nabla\hat{\Lambda}]^\top \nabla w \nabla\hat{\Lambda}[\nabla w]^\top - D(w\cdot\nabla\hat{\Lambda}) \Big).
\end{aligned}
$$

As already noted, thanks to the transverse motion on the fluid domain, to the fact that the trace of w on the moving boundary is colinear to e_2 and to the divergence free property of w, we have

$$
\int_{\mathcal{F}(t)} |Dw|^2 = \frac{1}{2}\int_{\mathcal{F}(t)} |\nabla w|^2.
$$

Consequently, we deduce that (141) holds true for any regular enough couple (w, b). Even though all the terms in the final identity are well-defined for vector-fields w satisfying (140) (we already noticed in Remark 1 that with the regularity (140) we have that $t \mapsto \int_{\mathcal{F}(t)} |\nabla w|^2(\cdot, t)$ is continuous) we need here to get more precise on the meaning of boundary terms in integration by parts. Note in particular that, with (140), we only have $\partial_t w \in L^2(\mathcal{Q}_t)$ so that its trace on $y = h(x,t)$ is not well-defined (though it is divergence-free).

There are several ways to overcome this difficulty. For instance, one may note that the computations above are completely rigorous if we assume further that $\nabla\partial_t w \in L^2(\mathcal{Q}_t)$. We may then end the proof by a density argument. Indeed, reproducing computations in the proof of Proposition 1 we have that the regularity (139)–(143), also satisfied by h, ensures that the transformation $w \mapsto \hat{w}$ defined by:

$$
w(x,y,s) = B_h^{-\top}\left(x, \frac{y}{h(x,s)}\right) \hat{w}\left(x, \frac{y}{h(x,s)}, s\right) \quad (x,y,s) \in \mathcal{Q}_t,
$$

realizes an homeomorphism between the set of w with regularity (140) and the set

$$
\mathcal{U} := H^1_\sharp(\Omega_1 \times (0,T)) \cap L^2((0,T); H^2_\sharp(\Omega_1)).
$$

Furthermore, this mapping exchanges the subset of w and $\hat{w}$ whose (space)-gradients are square integrable on the time/space domain. As $\mathcal{U} \subset C([0,T]; H^1_\sharp(\Omega_1))$, we note also that if $\hat{w}_n$ converges to $\hat{w}$ in $\mathcal{U}$ then $t \mapsto \int_{\mathcal{F}(t)} |\nabla w_n(\cdot,t)|^2$ converges to $t \mapsto \int_{\mathcal{F}(t)} |\nabla w(\cdot,t)|^2$ in $C([0,T])$. Consequently, we can pass to the limit in the identity (141) for a sequence of w_n such that the associated $\hat{w}_n$ converges in $\mathcal{U}$ to $\hat{w}$.

We can then decompose any $\hat{w} \in \mathcal{U}$ into

$$
\hat{w} = \hat{w}_0 + \tilde{U}[b] \qquad (\text{ with } \tilde{U} \text{ defined in the proof of Proposition 9 })
$$

and approximate b by considering its (L^2-)orthogonal projection on the first eigenmodes of the beam operator (∂_{xxxx}) and $\hat{w}_0$ by considering its (L^2-)orthogonal projection on the first eigenmodes of the Stokes operator with homogeneous Dirichlet boundary conditions on $z = 0$ and $z = 1$. □

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